

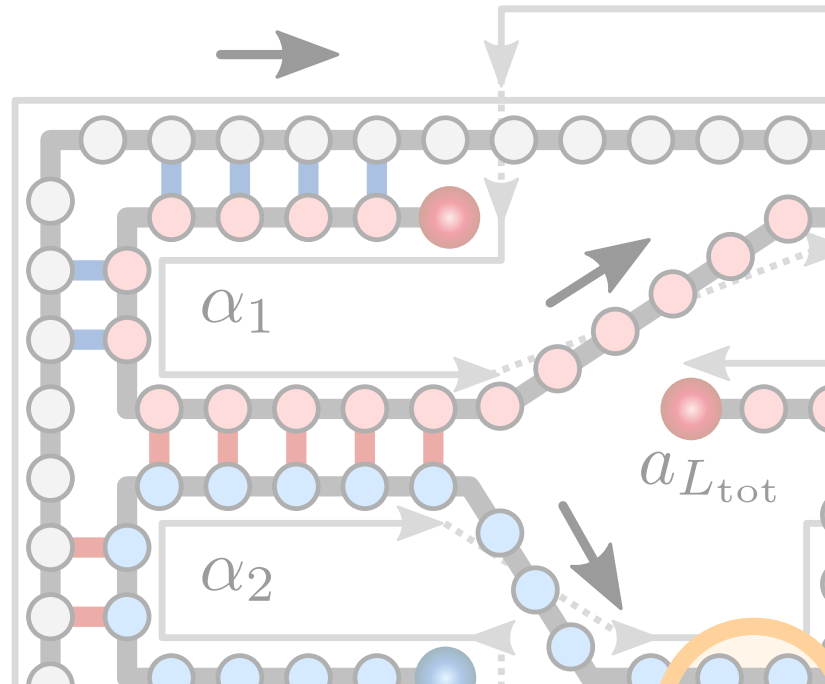
Topological states in a microscopic model of interacting fermions

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B8



Motivation

The Majorana Chain



The Majorana Chain # Setting

Open chain of spinless fermions:
 Nearest-neighbour hopping + Pairing + Chemical potential

$$H = - \sum_{i=1}^{L-1} \left[w a_i^\dagger a_{i+1} - |\Delta| a_i a_{i+1} + \text{h.c.} \right] - \mu \sum_{i=1}^L \left(a_i^\dagger a_i - \frac{1}{2} \right)$$



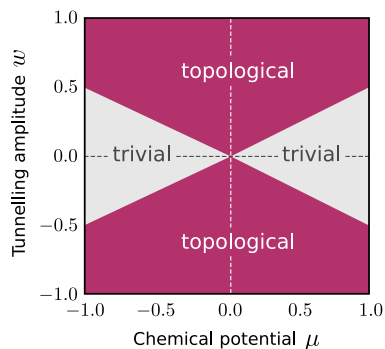
$$c_{2i-1} \equiv a_i + a_i^\dagger \quad \text{and} \quad c_{2i} \equiv i \left(a_i^\dagger - a_i \right) \quad \text{for } i = 1, \dots, L$$



Self-adjoint fermions
 (Majorana fermions)

$$\{c_l, c_m\} = 2\delta_{l,m} \quad \text{and} \quad c_l = c_l^\dagger$$

The Majorana Chain # Phases

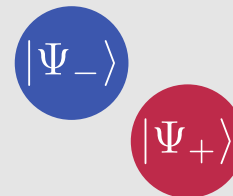


A Topological phase

Dominant hopping & pairing



Two degenerate ground states:



B Trivial phase

Dominant chemical potential

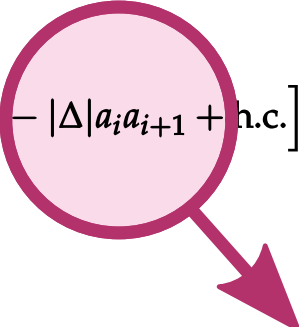


Unique ground state:



Beyond Mean Field Models?

$$H = - \sum_{i=1}^{L-1} \left[w a_i^\dagger a_{i+1} - |\Delta| a_i a_{i+1} + \text{h.c.} \right] - \mu \sum_{i=1}^L \left(a_i^\dagger a_i - \frac{1}{2} \right)$$



No particle-number conservation
(due to mean field origin)

Not justified in small, closed, number-conserving setups
(e.g., optical lattices)



Particle-number conserving theory?

Previous Research

→ M. Cheng and H.-H. Tu (2011). Physical Review B, 84(9), 094503.
Majorana edge states in interacting two-chain ladders of fermions.

→ J. D. Sau et al. (2011). Physical Review B, 84(14), 144509.
Number conserving theory for topologically protected degeneracy in one-dimensional fermions.

→ L. Fidkowski et al. (2011). Physical Review B, 84(19), 195436.
Majorana zero modes in one-dimensional quantum wires without long-ranged superconducting order.

→ J. Ruhman et al. (2015). Physical Review Letters, 114, 100401.
Topological States in a One-Dimensional Fermi Gas with Attractive Interactions.

→ C. V. Kraus et al. (2013). Physical Review Letters, 111(17), 173004.
Majorana Edge States in Atomic Wires Coupled by Pair Hopping.

→ G. Ortiz et al. (2014). Physical Review Letters, 113, 267002.
Many-body characterization of topological superconductivity: The Richardson-Gaudin-Kitaev chain.

Bosonization

Numerical (DMRG)

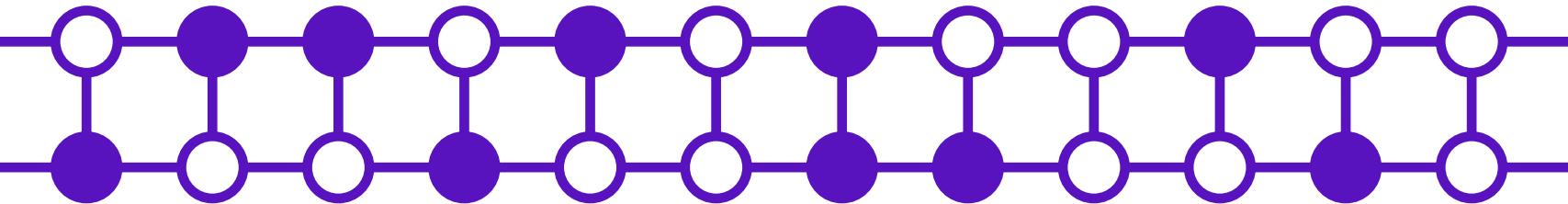
Long-Range



Short-range interacting theory
 Exact ground state(s)
 Majorana-like modes on edge

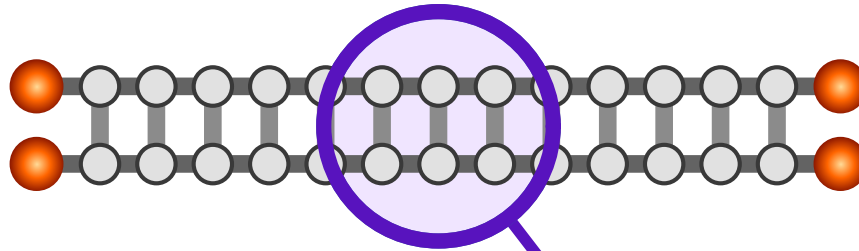
The Model

A Number-conserving Theory



Setting

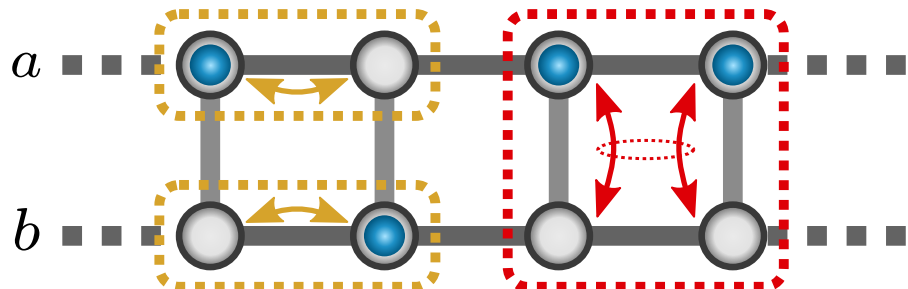
Open **DOUBLE** chain of spinless fermions ("ladder"):



Interacting Hamiltonian:

INTRAchain and **INTER**chain couplings

$$H = H^a + H^b + H^{ab}$$

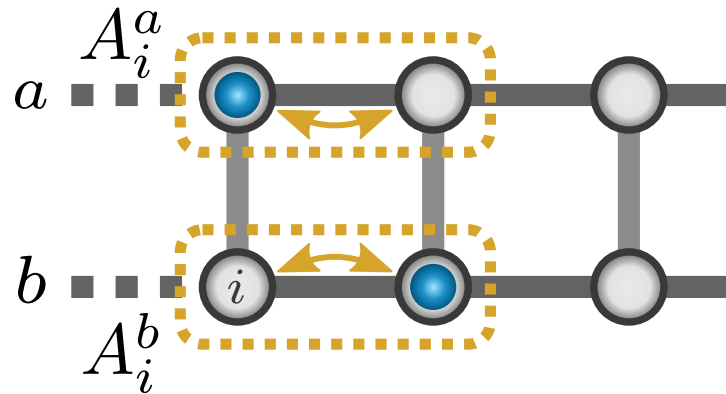


Intrachain Physics

$$H^a = \sum_{i=1}^{L-1} A_i^a (\mathbb{1} + A_i^a)$$

$$a_i a_{i+1}^\dagger + a_{i+1} a_i^\dagger$$

▶ Locally positive-semidefinite operators



Single particle hopping + density-density interactions :

$$H_i^a = \left[-a_i^\dagger a_{i+1} - a_{i+1}^\dagger a_i \right] + \left[n_i^a (1 - n_{i+1}^a) + n_{i+1}^a (1 - n_i^a) \right]$$

Interchain Physics

$$H^{ab} = \sum_{i=1}^{L-1} B_i (\mathbb{1} + B_i)$$

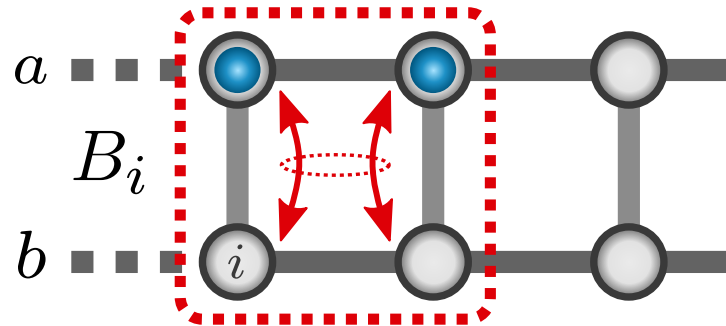
$$a_i^\dagger a_{i+1}^\dagger b_i b_{i+1} + b_i^\dagger b_{i+1}^\dagger a_i a_{i+1}$$

▶ Locally positive-semidefinite operators

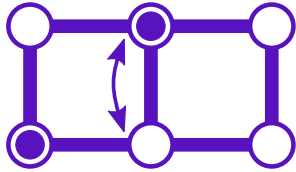
Pair hopping + Pair density-density interactions :

$$H_i^{ab} = -a_i^\dagger a_{i+1}^\dagger b_{i+1} b_i - b_i^\dagger b_{i+1}^\dagger a_{i+1} a_i$$

$$+n_i^a n_{i+1}^a (1 - n_i^b)(1 - n_{i+1}^b) + n_i^b n_{i+1}^b (1 - n_i^a)(1 - n_{i+1}^a)$$

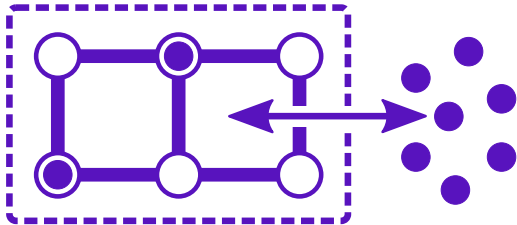


Symmetries



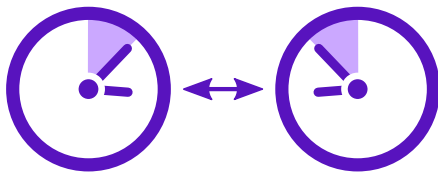
Subchain Parity

$$P_a = \prod_i (-1)^{n_i^a}$$



Total Particle Number

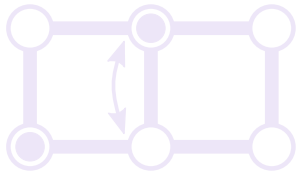
$$N = \sum_i (n_i^a + n_i^b)$$



Time Reversal

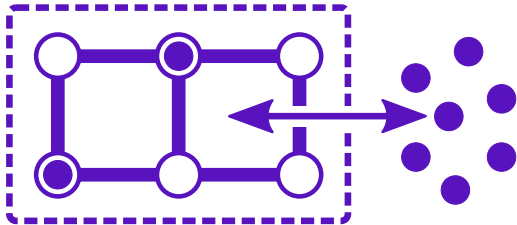
$$\mathcal{T} = (\bullet)^*$$

Symmetries



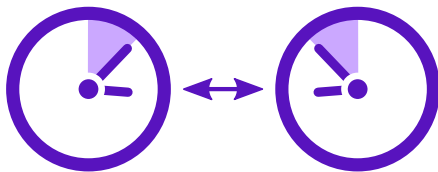
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Total Particle Number

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Time Reversal

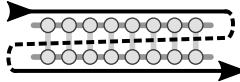
$$\mathcal{T} = (\bullet)^*$$

Results

Exact Ground States
& Braiding



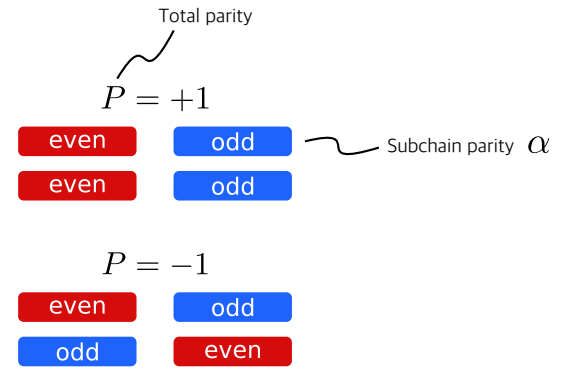
Exact Ground States

- ① Choose gauge (Fock basis) conveniently: 
- ② Exploit local positivity of Hamiltonian (frustration free)
- ③ Construct ground states rigorously from scratch ...
- ④ Two degenerate zero-energy ground states in each N sector:

Equal-weight superposition of all configurations with fixed total particle number & subchain parity.

$$|N, \alpha\rangle = \mathcal{N}_{L,N,\alpha}^{-1/2} \sum_{n, (-1)^n = \alpha} |n\rangle_a |N - n\rangle_b$$

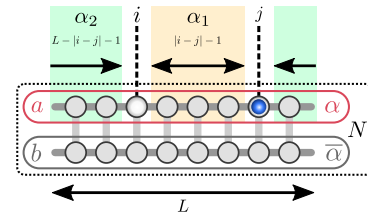
$$|n\rangle_x \equiv \sum_{|\mathbf{n}|=n} |\mathbf{n}\rangle_x$$



Evaluate all correlators & expectation values efficiently using combinatorics:

$$\langle N, \alpha | a_i^\dagger a_j | N, \alpha \rangle = \binom{L, L}{\alpha}_N^{-1} [\Lambda_{+1, -\alpha} - \Lambda_{-1, \alpha}]$$

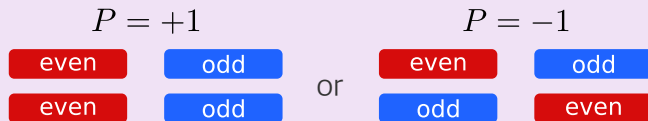
Generalised Binomial Coefficients



Exact Ground States # Comparison

N-conserving

- ▶ Two degenerate ground states in each N-sector:



- ▶ Zero-energy ground states:

Equal-weight superposition of all configurations with fixed total particle number & subchain parity.

$$(w = \Delta, \mu = 0)$$

Majorana Chain

- ▶ Two degenerate ground states:



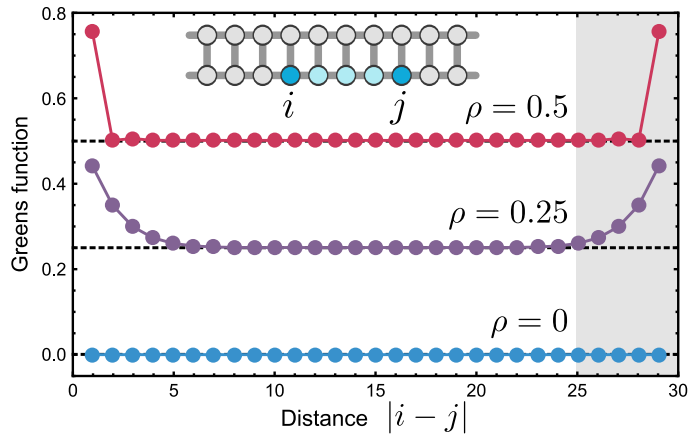
- ▶ Zero-energy ground states:

Equal-weight superposition of all configurations with fixed total parity.

Topological Properties # Edge States

INTRA-chain single-particle correlation

$$\langle a_i^\dagger a_j \rangle$$

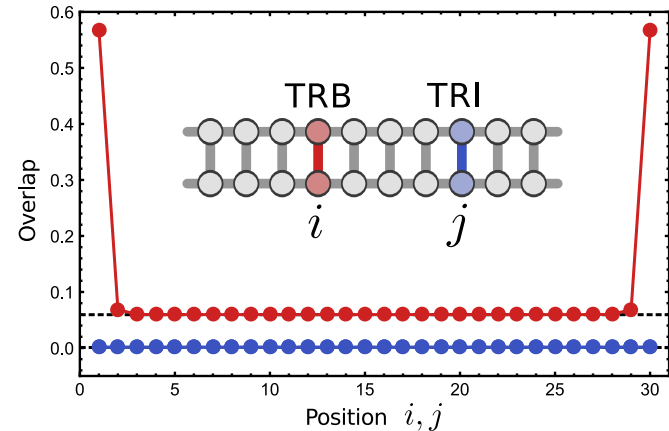


- ▶ Exponential decay away from edges ("Edge states")

INTER-chain single-particle hopping overlap

$$\text{TRI} : \langle -\alpha | a_\delta^\dagger b_\delta + b_\delta^\dagger a_\delta | \alpha \rangle \rightarrow 0$$

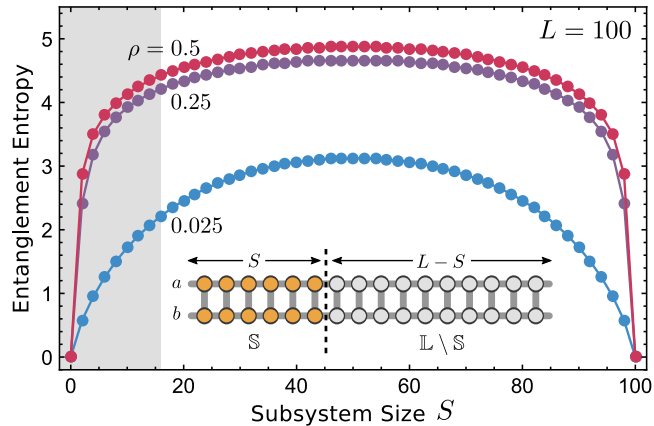
$$\text{TRB} : \langle -\alpha | i a_\delta^\dagger b_\delta - i b_\delta^\dagger a_\delta | \alpha \rangle \rightarrow e^{-\mu(\rho)\delta}$$



- ▶ Only time-reversal breaking (TRB) hopping lifts degeneracy at the edges

Topological Properties # Entanglement

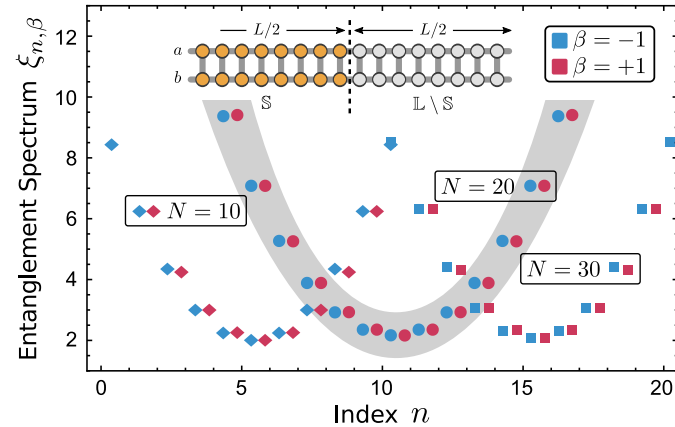
Entanglement Entropy



▶ Entanglement entropy for growing subsystem

▶ Logarithmic corrections to area law
→ gapless spectrum!

Entanglement Spectrum



▶ Entanglement spectrum for half-splitted double chain

▶ Stable two-fold degeneracy in thermodynamic limit

Excitations & Low-energy Physics

▶ Single-particle excitations:

Expectation:

Number-conservation → Gapless Goldstone modes

Symmetric superposition of single-magnon states:

$$|k; N, \alpha\rangle = P_1^a(k) \oplus P_1^b(k) |N, \alpha\rangle$$

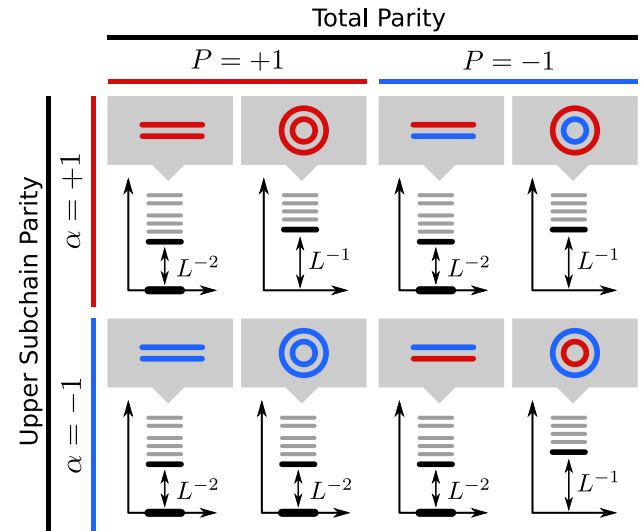
$$P_1^x(k) = \sum_{j=1}^L \cos\left[\frac{k}{2}(2j-1)\right] (-1)^{x_j^\dagger x_j}$$

Quadratic, gapless spectrum:

$$E_k = 4 \sin^2 \frac{k}{2}$$

$$k = m \frac{\pi}{L}, 0 \leq m < L$$

▶ Algebraically closing gap: L^{-2}



Braiding # Non-Abelian Statistics

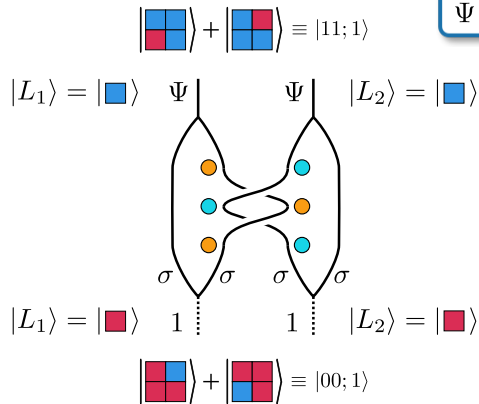
▶ Numerical evaluation of adiabatic time evolution:

Non-Abelian Holonomy:

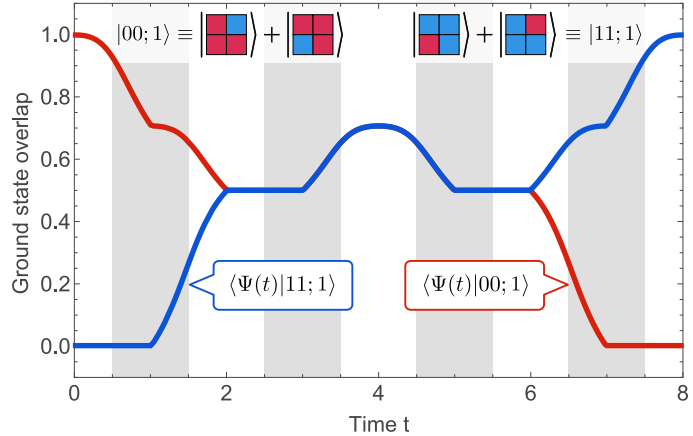
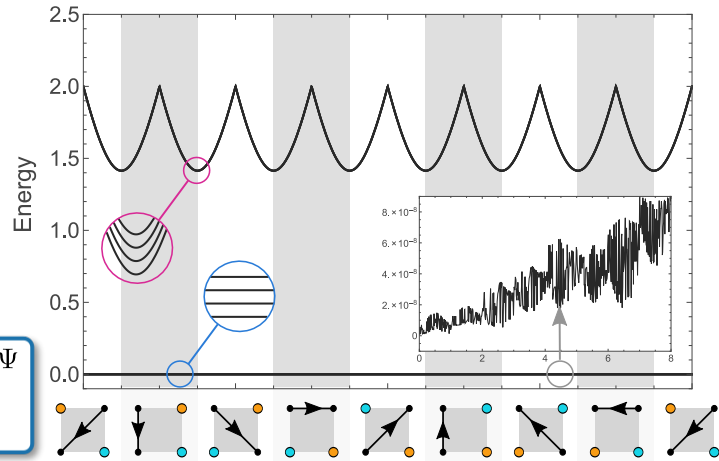
$$|11; 1\rangle = \exp \left[-i \int dt H_{\text{int}}(t) \right] |00; 1\rangle$$

Compare with Ising TQFT: $\{1, \sigma, \Psi\}$

$$\begin{aligned} \sigma \otimes \sigma &= 1 \oplus \Psi \\ \sigma \otimes \Psi &= \sigma \\ \Psi \otimes \Psi &= 1 \end{aligned}$$

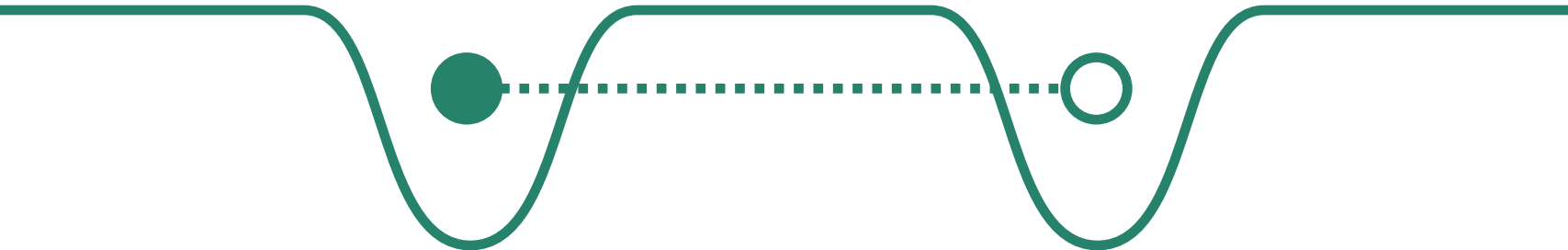


▶ Edge states = Ising anyons



Outlook

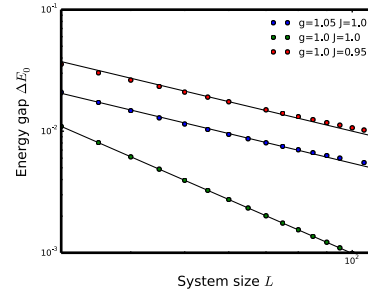
DMRG & more



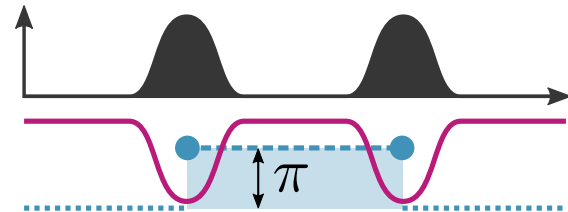
Current Research

Explore nearby phases by
MPS based DMRG simulations

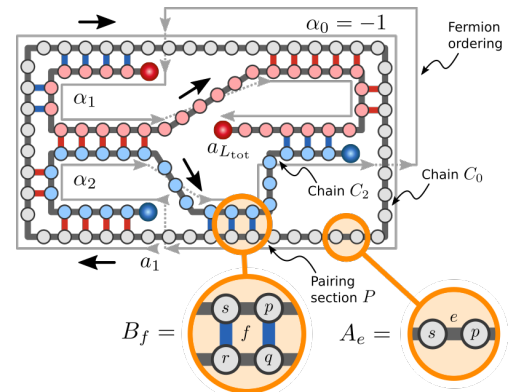
▶ Kai Guther, Masterthesis, ITP III

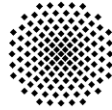


Explore low-energy sector by ansatz wave functions
(using the brain, not DMRG)



Explore more complex ladder networks
for braiding procedures





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funded by SFB TRR/21

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Questions?

Poster

1 Motivation

The Majorana Chain

1. Periodic model: Open chain of spinless fermions = "Majorana Chain"

$$H = \sum_{i=1}^L [\alpha c_i^\dagger c_{i+1} + \beta c_i^\dagger c_{i+1}^\dagger + \text{h.c.}]$$
2. Introduce Majorana fermions: "Real & Imaginary part of fermionic operators"

$$\gamma_i = c_i + c_i^\dagger, \tilde{\gamma}_i = i(c_i - c_i^\dagger)$$
3. Topological phase: Discrete hopping & pairing

$$H = \sum_{i=1}^L [\alpha \gamma_i \tilde{\gamma}_{i+1} + \beta \tilde{\gamma}_i \gamma_{i+1}]$$
4. Spectrum of Majorana chain: Stable zero energy edge modes

$$E = \pm \sqrt{\alpha^2 + \beta^2} \cos(k)$$
5. Applications: Why should we care?
 - Topological phase: Robustness against disorder
 - Majorana fermions: Non-Abelian quantum statistics
 - Non-Abelian quantum statistics: Braiding operations

→ Self adjoint fermions: $\gamma_i = \gamma_i^\dagger, \tilde{\gamma}_i = -\tilde{\gamma}_i^\dagger$

2 Exact Ground States

A Combinatorial Analysis

Observations:

1. Ground state is frustration-free
2. Local terms are positive semidefinite operators
3. Non-energy state = Ground state

Formalism ordering:

1. Fermion ordering
2. Ground state is frustration-free
3. Local terms are positive semidefinite operators
4. Non-energy state = Ground state

Example: Single-particle correlation

1. Split system according to fermion ordering
2. Reconfigurations with fixed subsector parity
3. Sum of two PfSKs: $\langle \gamma_i \tilde{\gamma}_j \rangle = \frac{1}{2} (\langle \gamma_i \tilde{\gamma}_j \rangle + \langle \tilde{\gamma}_i \gamma_j \rangle)$

Derive ground state properties:

1. Frustration-free expectation values = Counting of configurations
2. Define Pauli-matrix Exponential Coefficients (PECO)
3. Example: Normalizing factor: $\langle \mathbb{1} \rangle = \text{Tr}[\rho]$

Edge states:

1. Introduce single-particle correlation
2. Exponential decay away from edges (Edge states)

Ground state overlap:

1. Introduce single-particle hopping overlap
2. Only time-reversal breaking (TRB) hopping lifts degeneracy at the edges

Entanglement entropy (EE):

1. EE for ground subsystem
2. Logarithmic corrections to area law = g/4π

Entanglement spectrum (ES):

1. ES for half-filled chiral chain (one mode)
2. Stable level-like degeneracy in thermodynamic limit

Our Model: Open Majorana chain with α and β hopping and pairing.

Non-Abelian degeneracy (logarithmic) ground state for open (closed) chiral chain.

Topological Properties: Degeneracy & Entanglement

Summary: Topological ground states with edge localization

3 Braiding Edge States

Non-Abelian Statistics

Setup for braiding edge states:

1. Allow for two localized edge states, where each other (then exchange sites)
2. Consider it weakly coupled chains L_1 and L_2 with a common "bulk" chain
3. In the quantum pump $n = 1, 2, \dots$
4. 3-dimensional low-energy Hilbert space

Non-Abelian braiding statistics of edge states (associated with $\pi/8$ phases)

4 Excitations

Gapless Goldstone Modes

Single-particle excitations:

1. Excitation: Number conservation & Gauging EE
2. Symmetric superposition of single-magnon states: $\rho(X) = \frac{1}{2} (\rho(X) + \rho(X^\dagger))$
3. Gapless Goldstone modes: $\rho(X) = \frac{1}{2} (\rho(X) + \rho(X^\dagger))$
4. Quasiparticle spectrum of single chain to describe chain setup and arbitrary filling: $\rho(X) = \frac{1}{2} (\rho(X) + \rho(X^\dagger))$

Summary: Gap closing

1. OBC: $\rho(X)$ is scaling due to strongly anisotropic single-magnon states
2. PBC: $\rho(X)$ is scaling in odd odd sectors & $\rho(X)$ is scaling in the others
3. Algebraically filling gap with exponentially degenerate ground states
4. Duality Matrix Renormalization: Use MPS-based DMRG to compute GS energies for periodic setup
5. Verify exact results: $\rho(X)$ in odd odd sector and in odd odd sector for PBC
6. Numerical results: gap closing for PBC in even odd-even sectors

Approximately closing gap due to Goldstone modes in a confinement of normal, unconfined