Topological Quantum Error Correction: The Toric Code

PI3 Seminar

May 8, 2015

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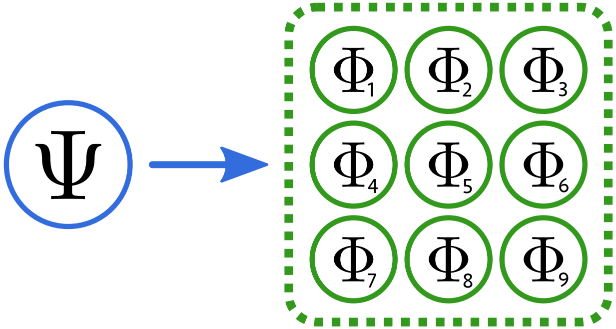
Outline

1. Introduction

→ (Quantum) Error correction

→ The Stabilizer formalism

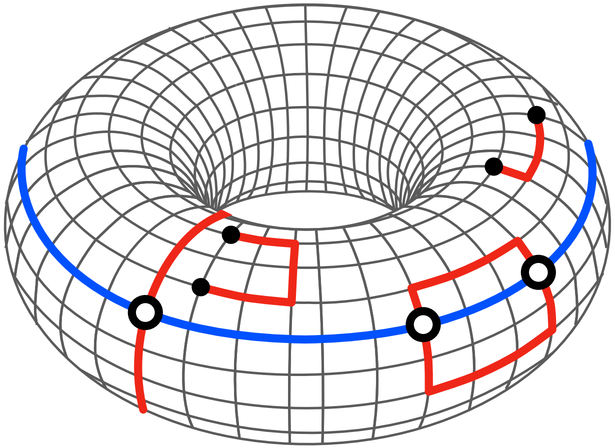


2. The Toric Code

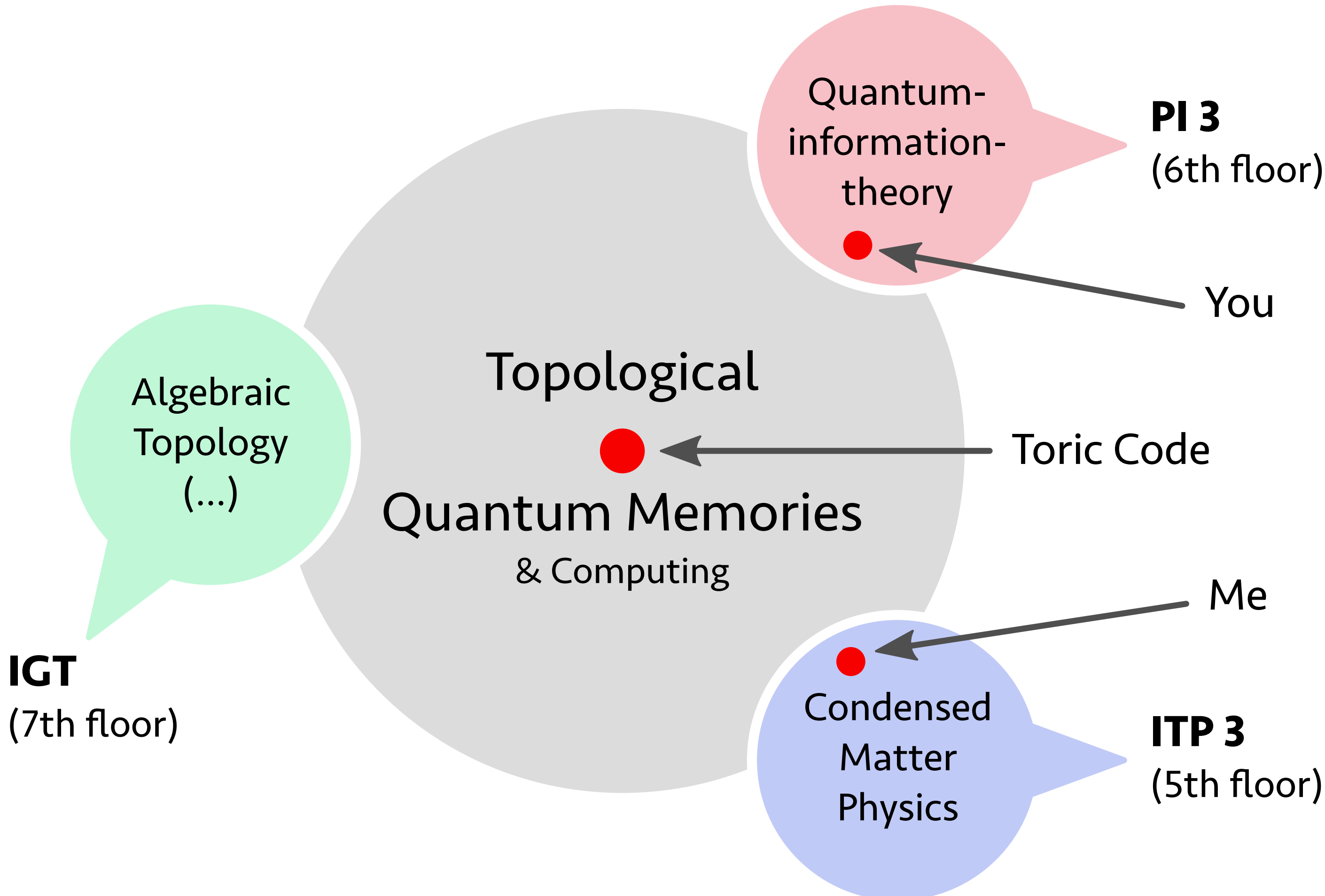
→ Rigorous construction

→ Error detection & correction

→ Experimental results



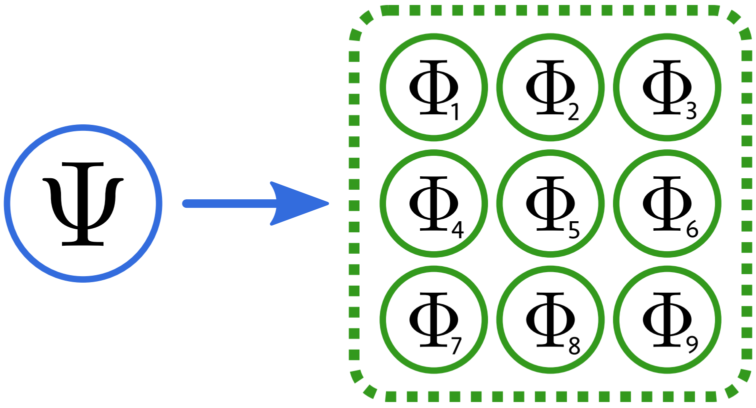
The big picture



1

Introduction

Quantum error correction and the Stabilizer formalism



(Quantum) Error correction

General idea:

→ Distribute few logical (qu)bits onto many physical (qu)bits

→ Hope: Errors do not "reach" logical (qu)bits

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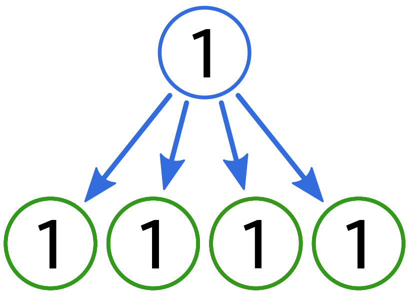
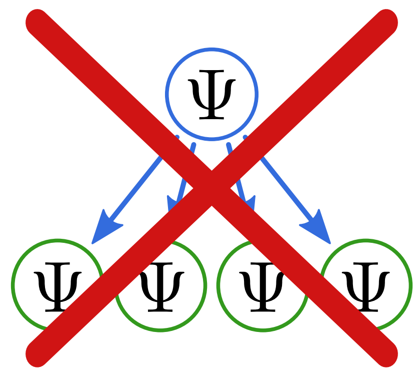
Classical



("Repetition Code")

Quantum



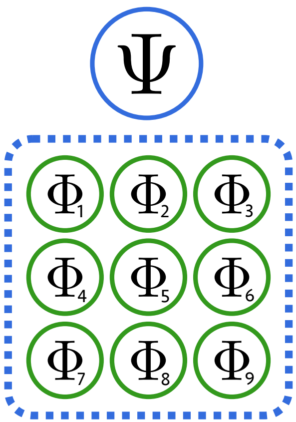
→ Clone bits
→ Arbitrary measurements

→ No-cloning Theorem
→ Restricted measurements

(Quantum) Error correction

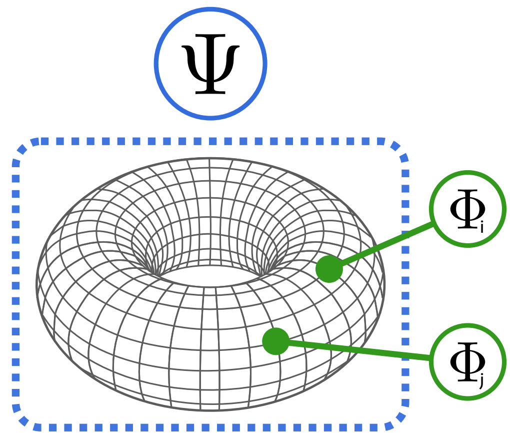
Solution: Quantum Codes & Quantum Error Correction

"The first one"



(Shor Code, 1995)

"The exciting one"



(Toric Code, 1997)

Topological QEC

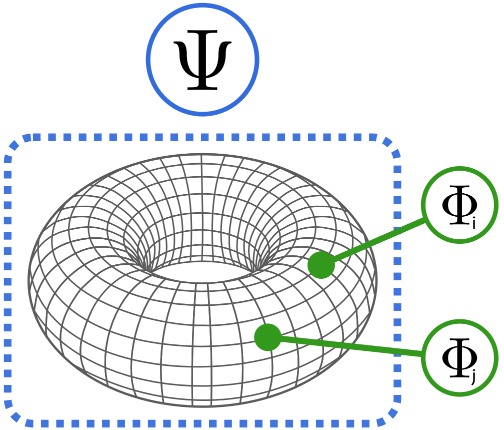
Self-Correction

A. Y. Kitaev, "Fault-tolerant quantum computation by anyons", Annals of Physics 303, 2-30 (2003)

(Quantum) Error correction

Solution: Quantum Codes & Quantum Error Correction

"The exciting one"



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Topological QEC

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The stabilizer formalism

Goal:

→ Describe highly entlangled many-qubit states

Problem:

→ Exponentially large Hilbert space:

# of complex variables needed

$$N \text{ qubits} \longrightarrow \mathcal{H} = \bigotimes_{i=1}^{N} \mathbb{C}_i^2 \longrightarrow \dim \mathcal{H}^N = 2^N$$

Solution:

→ Use group theoretical methods

→ Describe stabilizer group instead of the stabilized state

The stabilizer formalism

Pauli Group

$$G_N := \text{span}\left\{\mathbb{1}_1 \otimes \cdots \otimes \sigma_k^i \cdots \otimes \mathbb{1}_N \,|\, i \in \{x, y, z\}; 1 \leq k \leq N, k \in \mathbb{N}\right\}$$

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Stabilizer

independent & commuting generators

Choose Generator: $\mathcal{G} = \{g_i\}_{i=1}^{d} \subseteq G_N$

Define: $\mathcal{S} := \text{span}\,\mathcal{G}$ and $\mathcal{PS} := \left\{|\Phi\rangle \in \mathcal{H}^N \,|\, \mathcal{S}\,|\Phi\rangle = |\Phi\rangle\right\}$

Stabilizer (group)

Stabilized subspace (state)

The stabilizer formalism

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Stabilizer (group)

Stabilized subspace (state)

Dimension & Rank

$\mathcal{PS} \leq \mathcal{H}^N$ & $d = \text{rank}\,\mathcal{S}$ $\rightarrow$ $\dim \mathcal{PS} = 2^{N-d}$

The stabilizer formalism

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Dimension & Rank

$\mathcal{PS} \leq \mathcal{H}^N$ & $d = \text{rank}\,\mathcal{S}$ $\rightarrow$ $\dim \mathcal{PS} = 2^{N-d}$

→ $\mathcal{S}$ abelian subgroup of G_N

→ Describe $\mathcal{PS}$ in terms of $\mathcal{G}$

→ Not all states are stabilizer states

Example: $N = 2$, $\mathcal{G} = \{\sigma_1^x\sigma_2^x, \sigma_1^z\sigma_2^z\} \subseteq G_2$

$d = 2$

$[\sigma_1^x\sigma_2^x, \sigma_1^z\sigma_2^z] = 0$

$\dim \mathcal{PS} = 2^{2-2} = 1$

$$\mathcal{PS} = \left\{|EPR\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)\right\}$$

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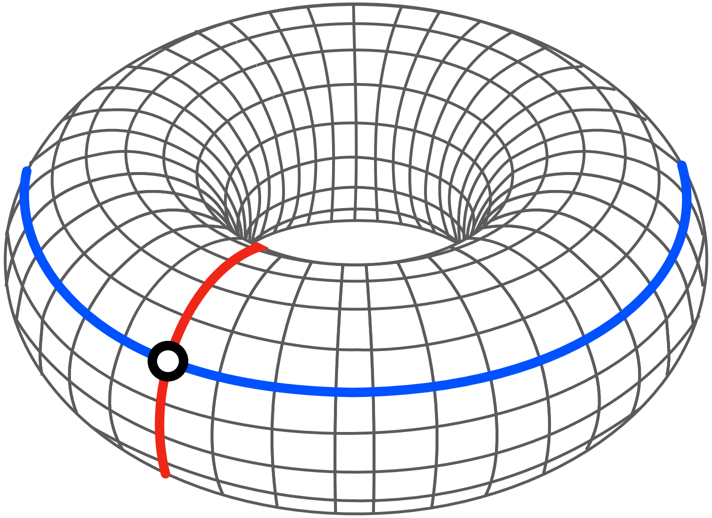
The Toric Code

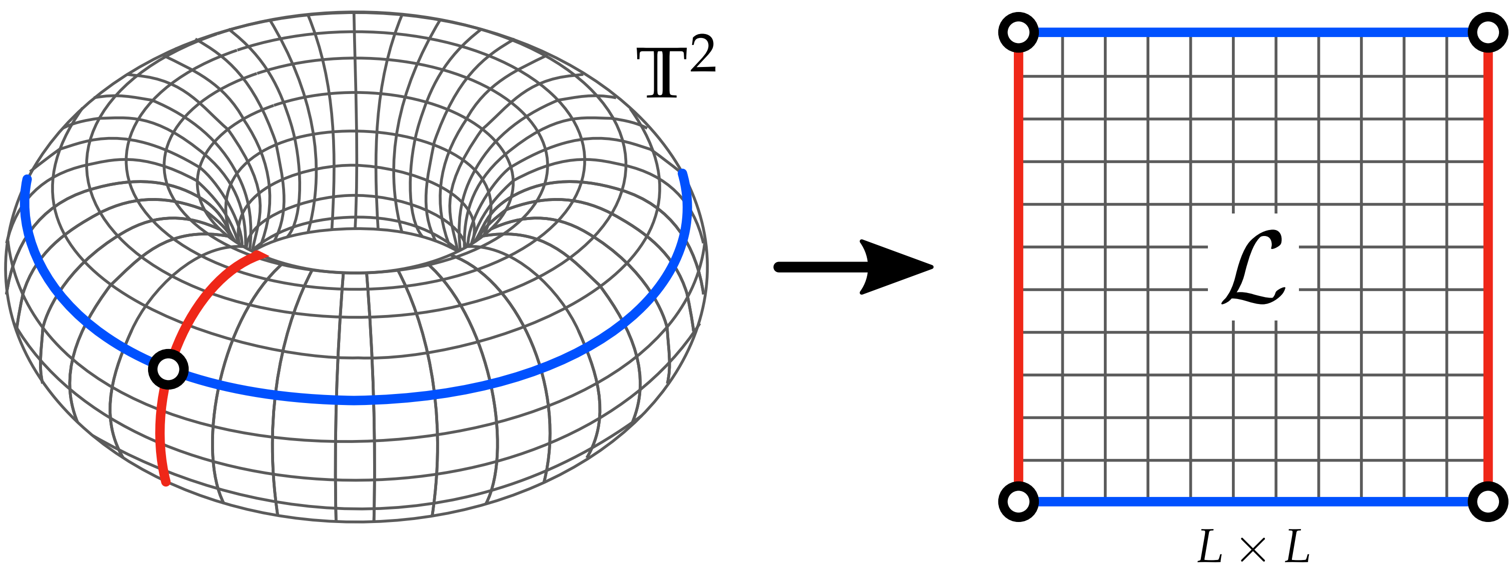
A rigorous introduction



The Setting

1. Define the topology

Square lattice with periodic boundary conditions embedded into the torus:



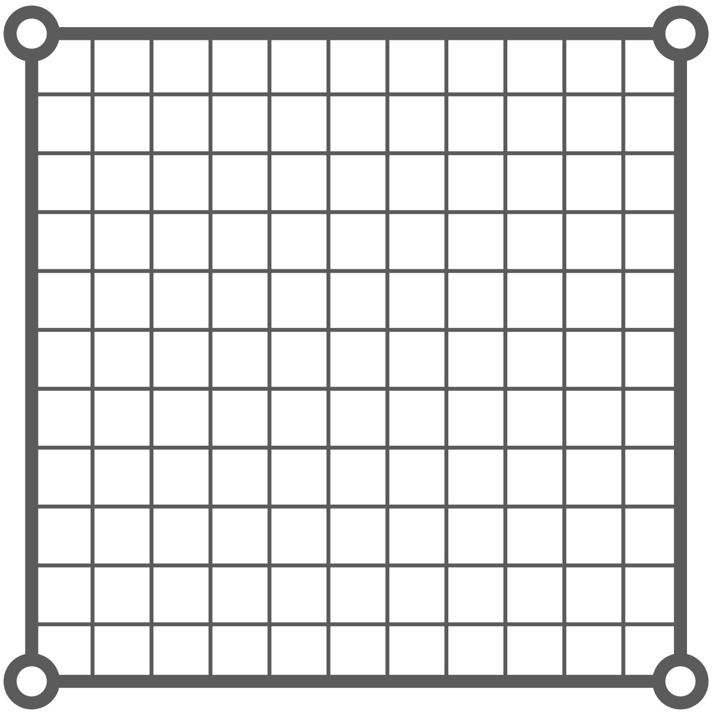
The lattice defines **vertices** and **edges**

$$|\mathbb{V}(\mathcal{L})| = L^2 \qquad |\mathbb{E}(\mathcal{L})| = 2L^2$$

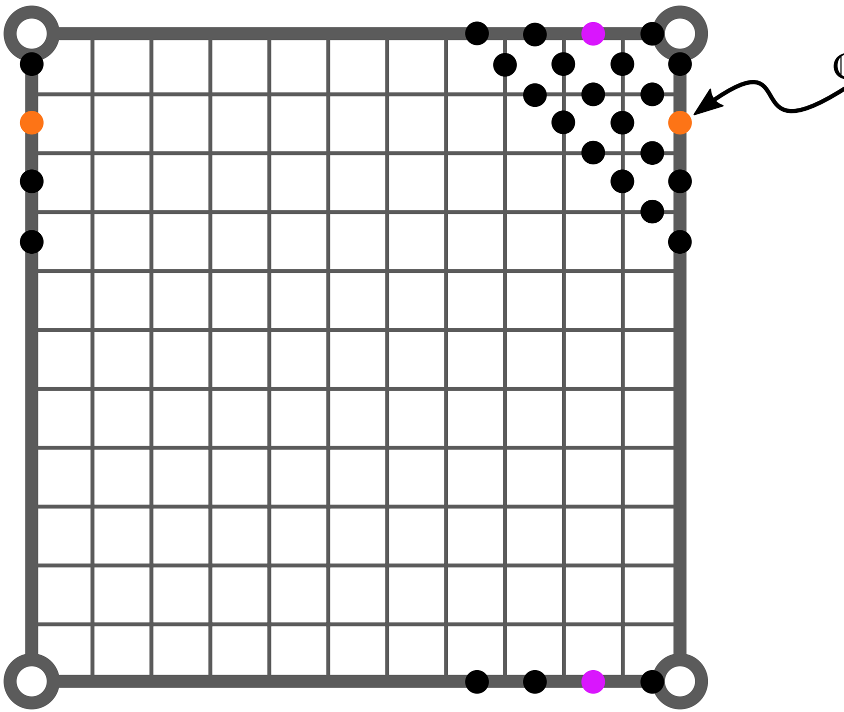
The embedding defines **faces/plaquettes**

$$|\mathbb{P}(\mathcal{L})| = L^2$$

The Stabilizer



The Stabilizer

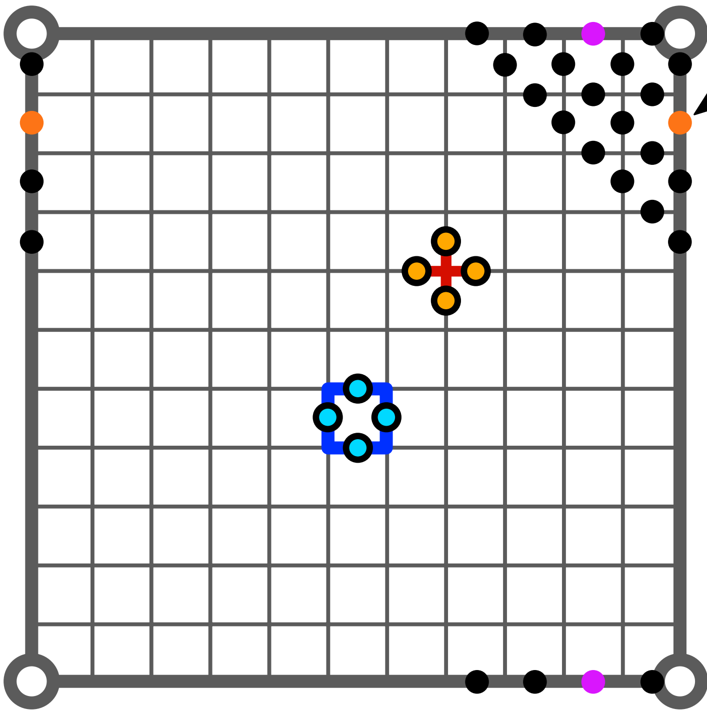


2. Define the Hilbert space

Attach spins/qubits to all edges:

$\mathbb{C}_e^2,\, e \in \mathbb{E}\,(\mathcal{L})$ ⟶ $\mathcal{H}(\mathcal{L}) = \bigotimes_{e\in\mathbb{E}(\mathcal{L})} \mathbb{C}_e^2$

The Stabilizer



2. Define the Hilbert space

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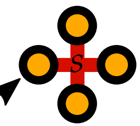
$$\mathbb{C}^2_e,\ e \in \mathbb{E}(\mathcal{L}) \longrightarrow \mathcal{H}(\mathcal{L}) = \bigotimes_{e\in\mathbb{E}(\mathcal{L})} \mathbb{C}^2_e$$

3. Define the stabilizer

A star operator per vertex, ...

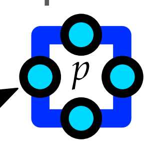
$$A_s := \prod_{i\in s} \sigma_i^x$$



$\in \mathbb{V}(\mathcal{L})$

... a plaquette operator per face, ...

$$B_p := \prod_{i\in p} \sigma_i^z$$

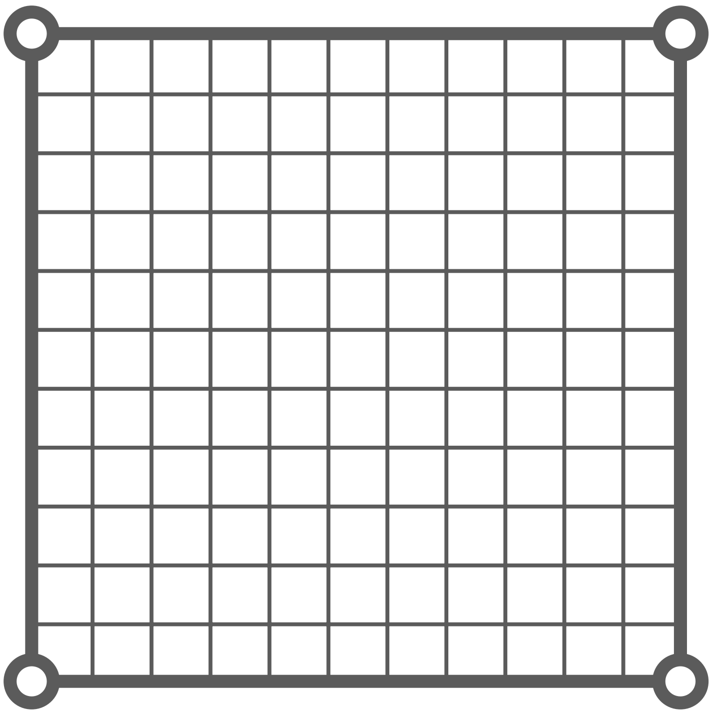


$\in \mathbb{P}(\mathcal{L})$

... and the stabilizer group:

$$\mathcal{S} := \text{span}\left\{A_s, B_p\right\}_{s\in\mathbb{V}(\mathcal{L}),p\in\mathbb{P}(\mathcal{L})}$$

The Stabilizer



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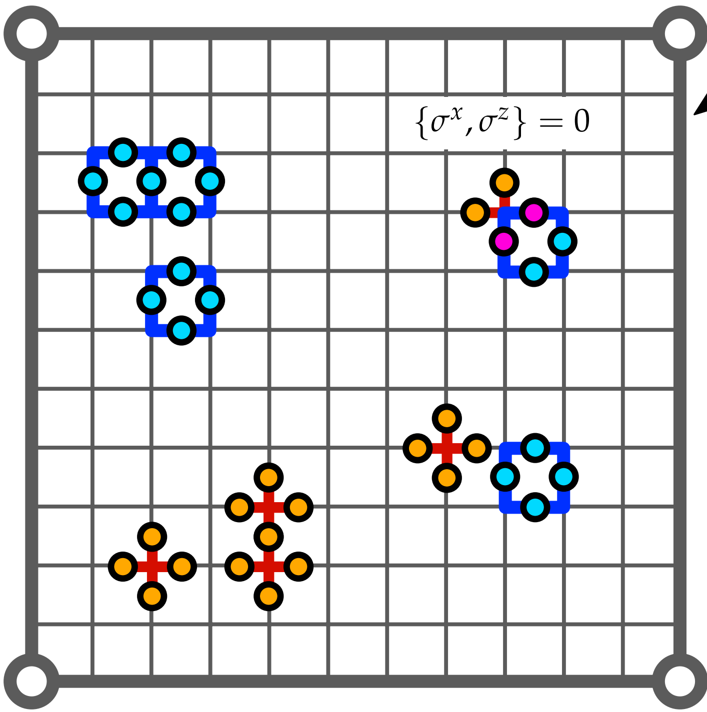


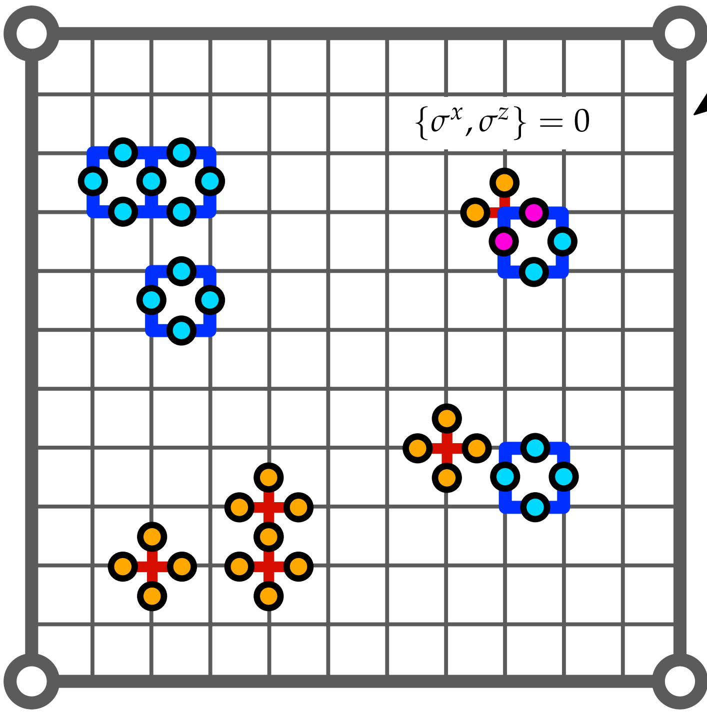
Is this a stabilizer?

All star- and plaquette operators commute:

$$[A_s, A_{s'}] = \left[B_p, B_{p'}\right] = [A_s, B_p] = 0$$

→ Yes!

The Stabilizer



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Dimension of stabilized subspace?

Two non-trivial relations:

$$\prod_{s \in \mathbb{V}(\mathcal{L})} A_s = \mathbb{1} = \prod_{p \in \mathbb{P}(\mathcal{L})} B_p$$

Therefore the dimension is:

# of faces, # of sites

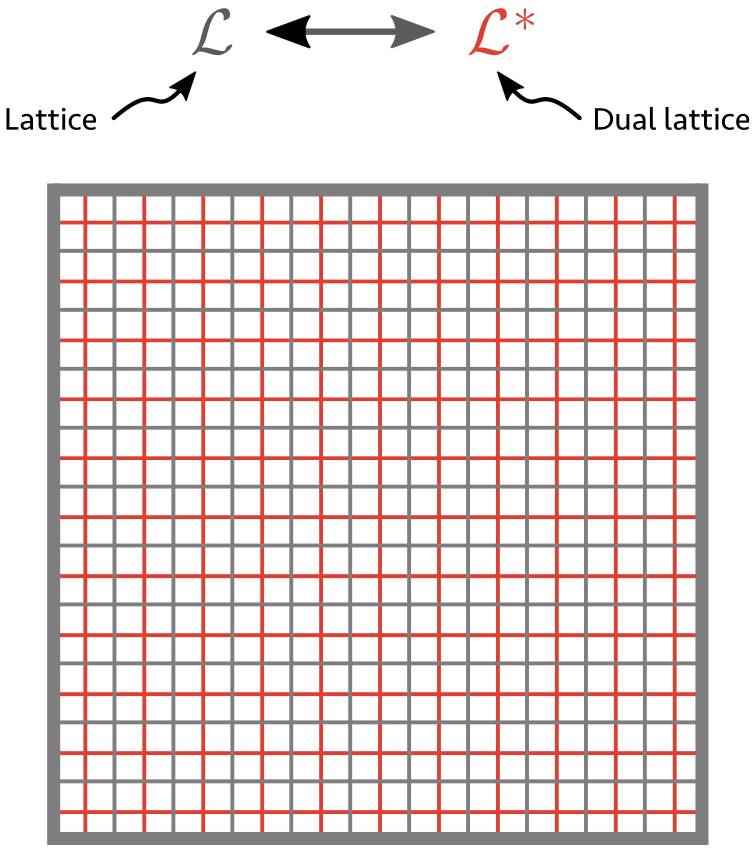
$$\operatorname{rank} \mathcal{S} = (L^2 - 1) + (L^2 - 1) = 2L^2 - 2$$

→ $$\dim \mathcal{PS} = 2^{2L^2 - (2L^2 - 2)} = 2^2 = 4$$

# of physical qubits, Stabilizer rank

Encode two logical qubits in $\mathcal{PS}$!

Dual Lattice & Strings



Construction

1. Faces of $\mathcal{L}$ become vertices of $\mathcal{L}^*$
2. Vertices of $\mathcal{L}$ become faces of $\mathcal{L}^*$
2. Edges of $\mathcal{L}$ remain edges of $\mathcal{L}^*$

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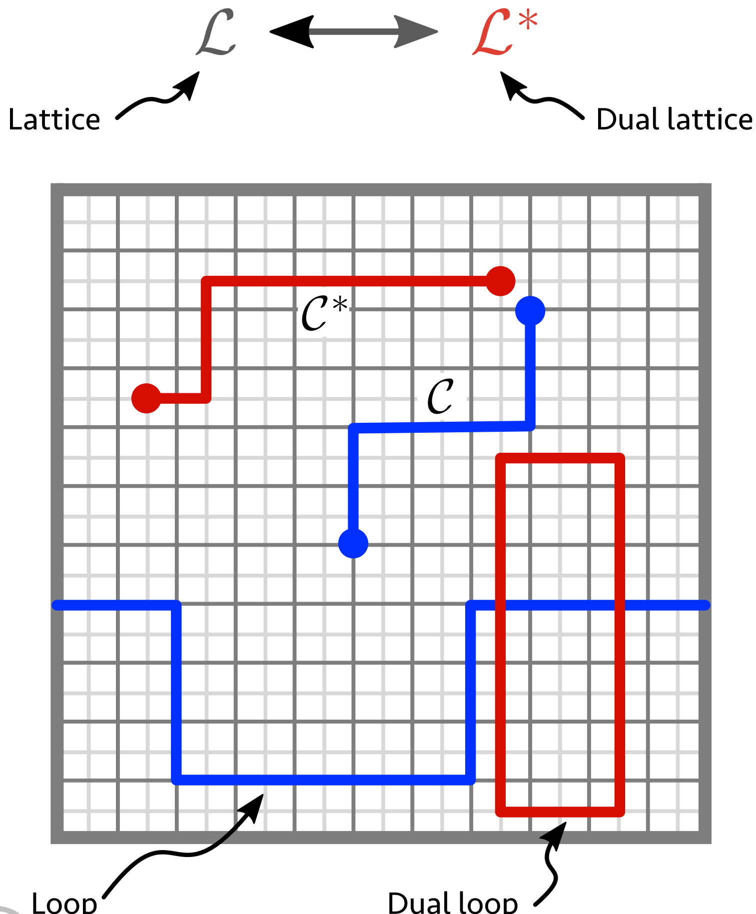

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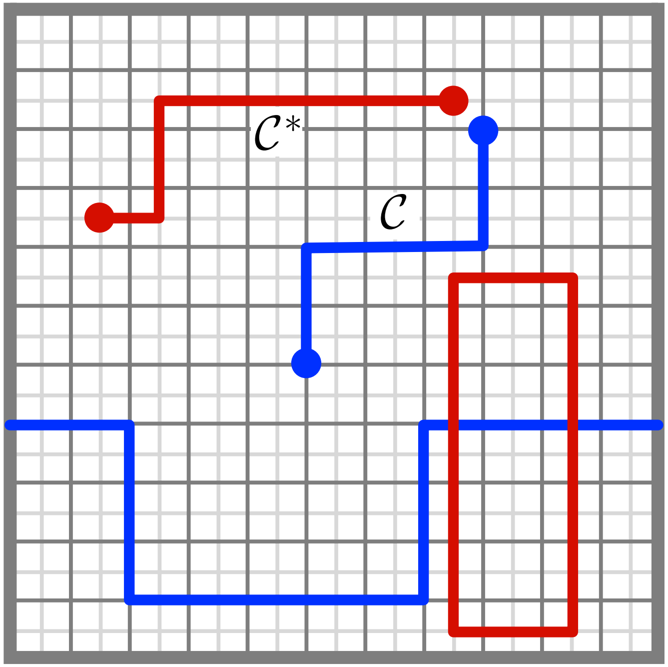
Strings & Loops

$\mathcal{C}$ = Open string on $\mathcal{L}$

$\mathcal{C}^*$= Open dual string on $\mathcal{L}^*$

Closed strings = Loops

String & Loop operators

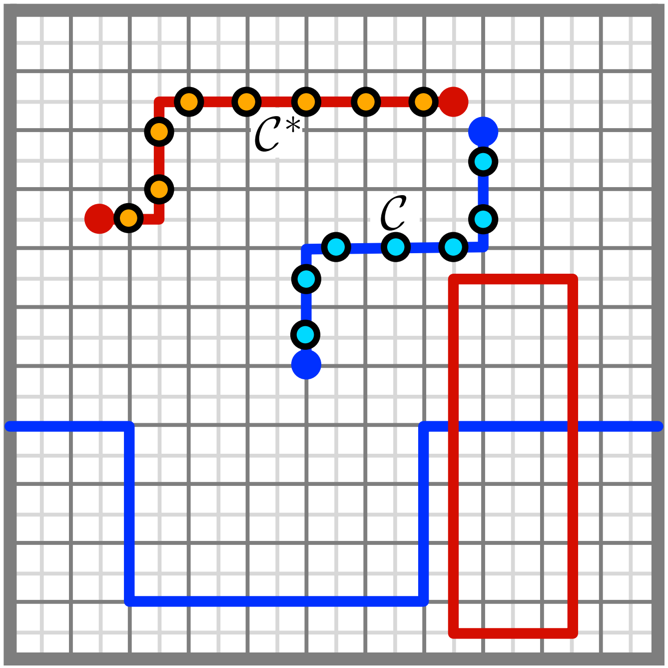


String & Loop operators

String & Loop operators

Representation on $\mathcal{H}(\mathcal{L})$:

$$X\left[\mathcal{C}^*\right] := \prod_{i^* \in \mathcal{C}^*} \sigma^x_{i^*}, \quad Z\left[\mathcal{C}\right] := \prod_{i \in \mathcal{C}} \sigma^z_i$$



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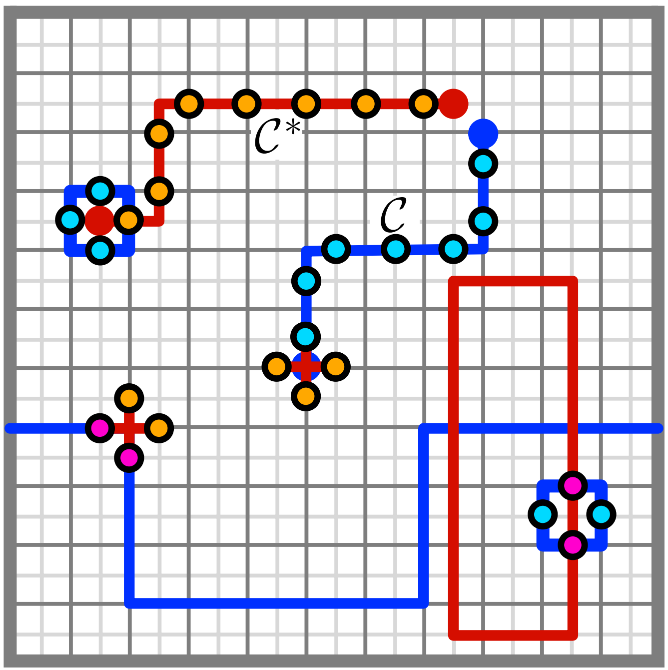


Commutation relations

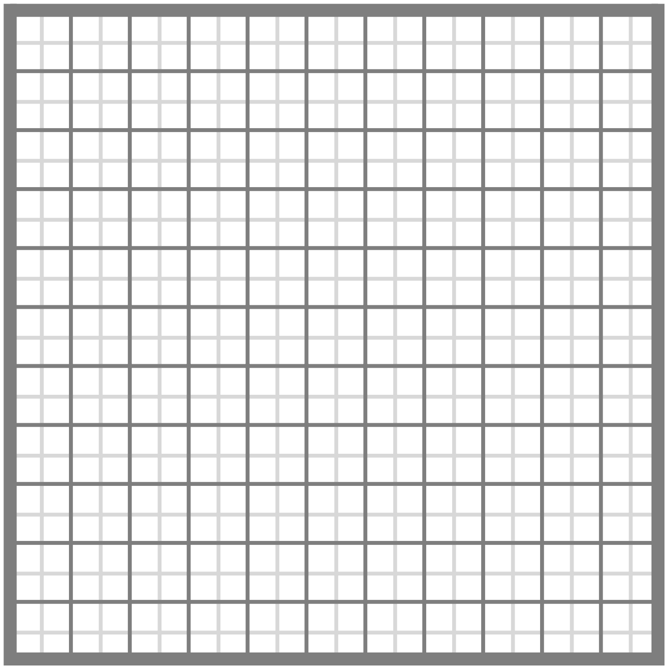
Stabilizers commute with strings except for their endpoints!

$$\left[A_s, X\left[\mathcal{C}^*\right]\right] = \left[B_p, X\left[\mathcal{C}^*\right]\right] = 0$$
$$\left[A_s, Z\left[\mathcal{C}\right]\right] = \left[B_p, Z\left[\mathcal{C}\right]\right] = 0$$

→ Loop operators commute with all stabilizers!

→ Loop operators are diagonalizable over $\mathcal{PS}$

The Code Space

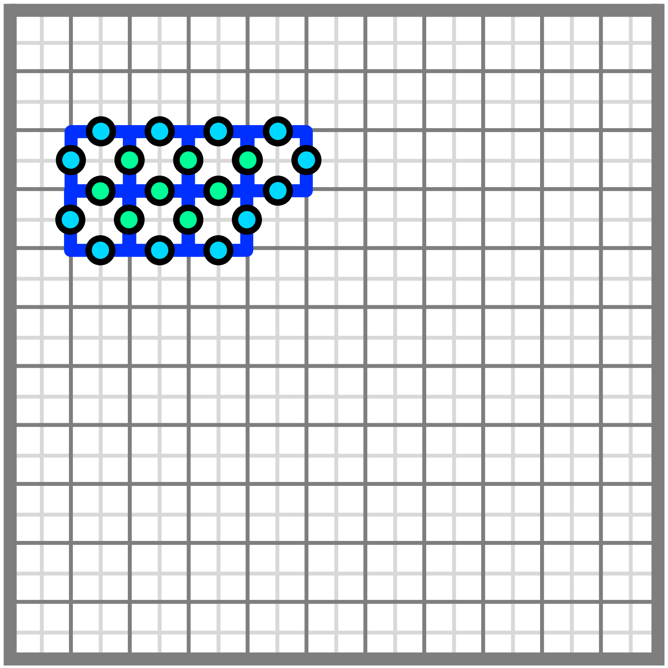


The Code Space

Trivial Loops:

→ Boundaries of plaquettes/stars

→ Act as identity on $\mathcal{PS}$



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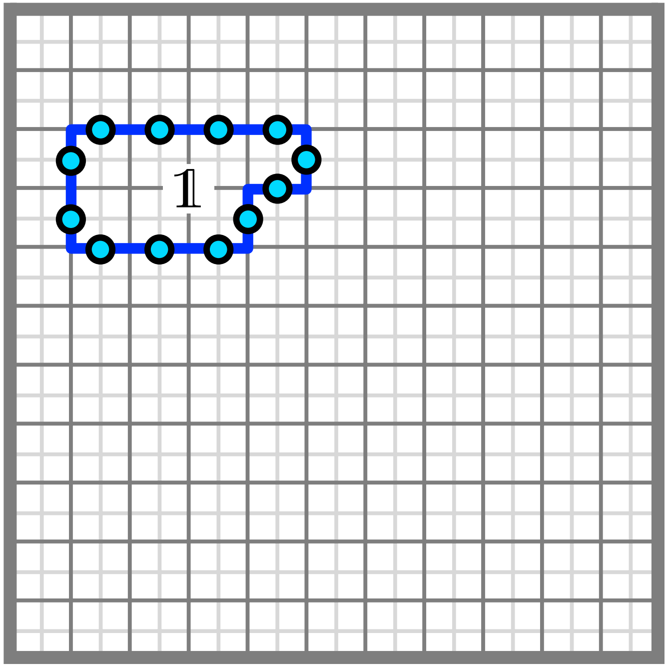

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Trivial Loops:

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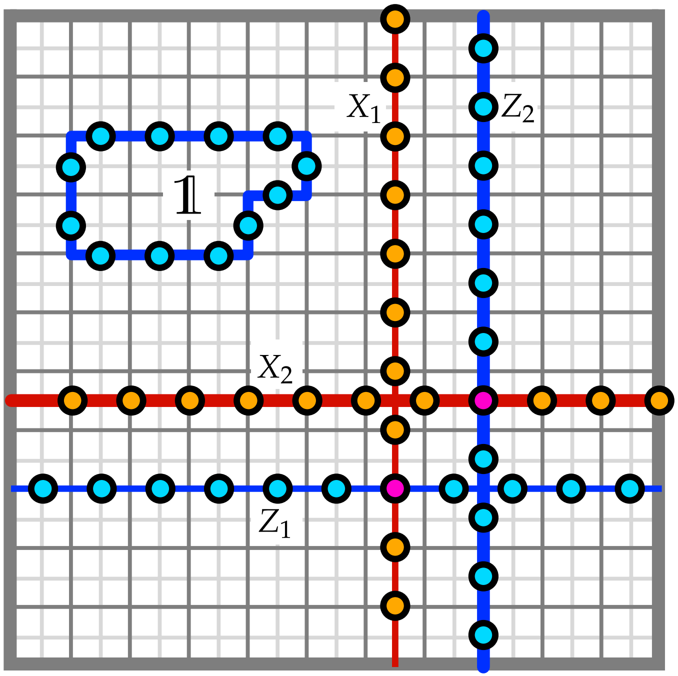
Non-Trivial Loops:

→ Wrap around the torus on $\mathcal{L}$ or $\mathcal{L}^*$

→ Four such operators: X_1, X_2, Z_1, Z_2



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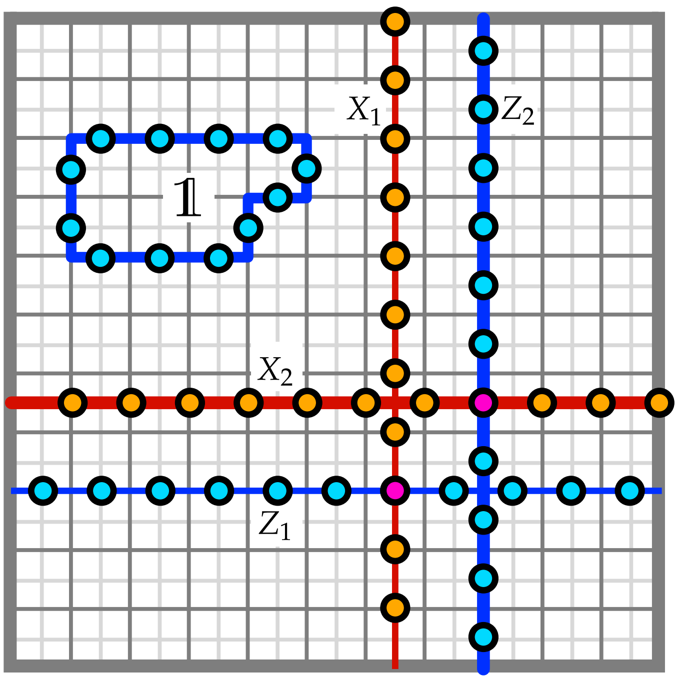

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Algebra

$$[Z_i, X_j] = [X_i, X_j] = [Z_i, Z_j] = 0$$
$$\{Z_i, X_i\} = 0\,, \quad i \neq j$$

→ Pauli algebra of 2 qubits on $\mathcal{PS}$

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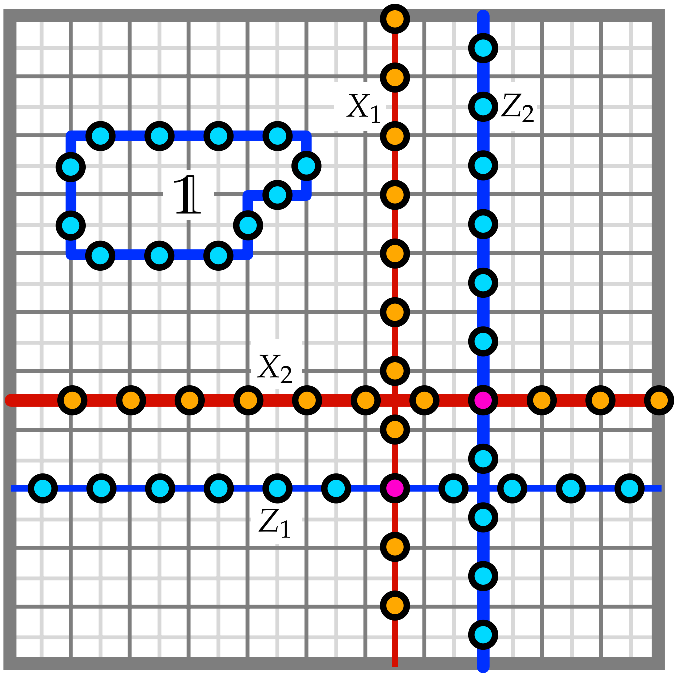

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→ Z_1 and Z_2 commute

Topological Quantumnumbers

Eigenbasis

$$Z_j\,|v_1, v_2\rangle = v_j\,|v_1, v_2\rangle\,, \quad v_j \in \{-1, 1\}$$

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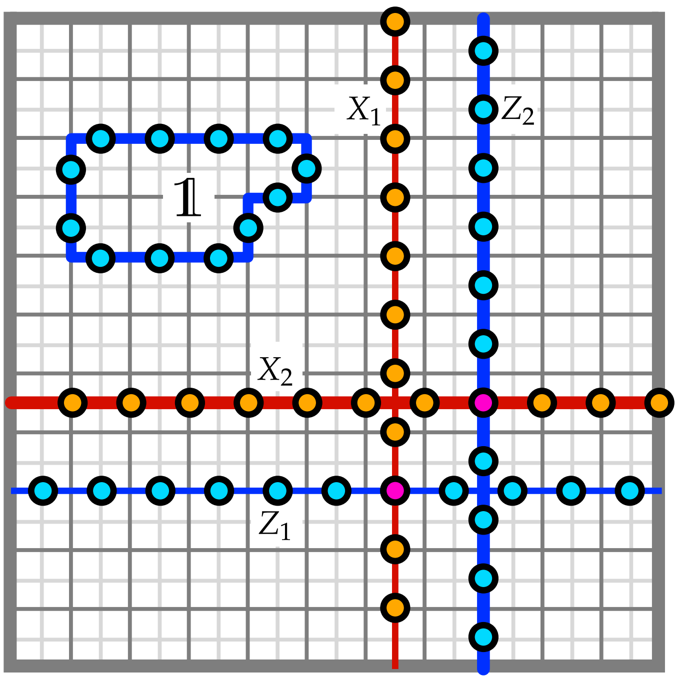

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$$Z_j\,|v_1, v_2\rangle = v_j\,|v_1, v_2\rangle\,, \quad v_j \in \{-1, 1\}$$

→ Action of X_1 and X_2 :

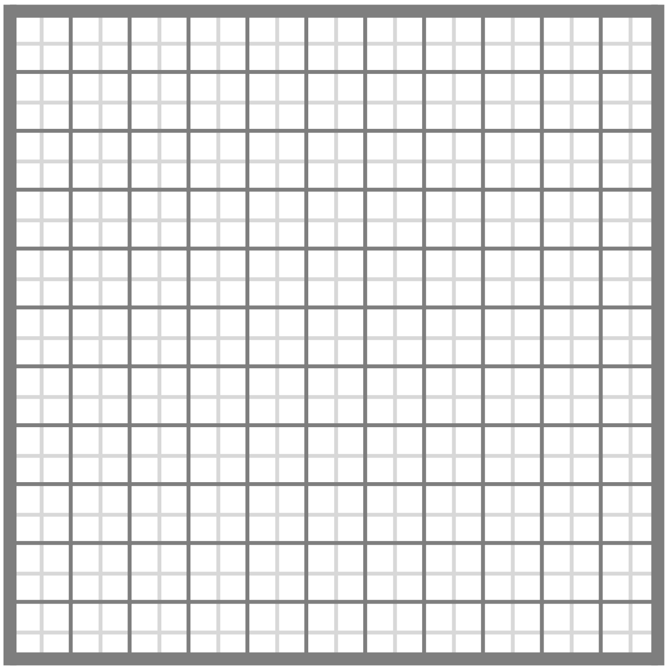
$$X_j\,|v_1, v_2\rangle = \begin{cases} |-v_1, v_2\rangle\,, & j = 1 \\ |v_1, -v_2\rangle\,, & j = 2 \end{cases}$$

Error detection

1. Initial state

→ System in the code space $\mathcal{P}\mathcal{S}$

→ Here: $|v_1, v_2\rangle = |\,1\,,\,1\,\rangle$



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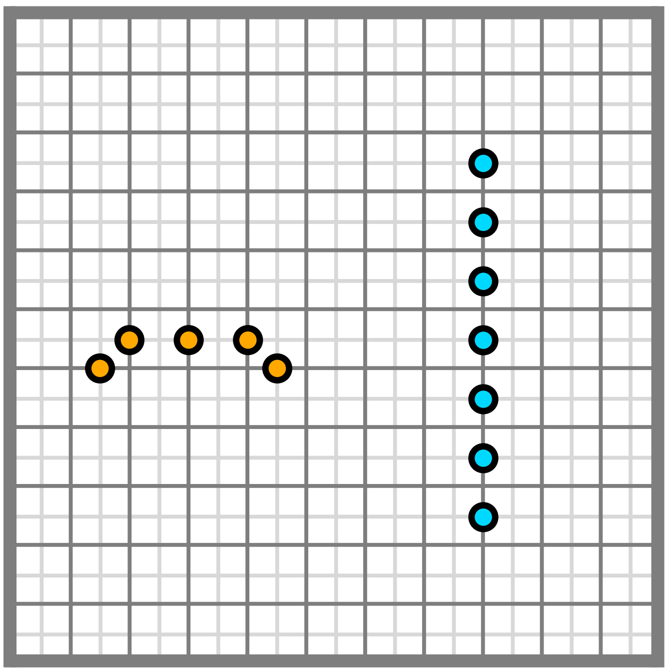


1. Initial state

→ System in the code space $\mathcal{PS}$

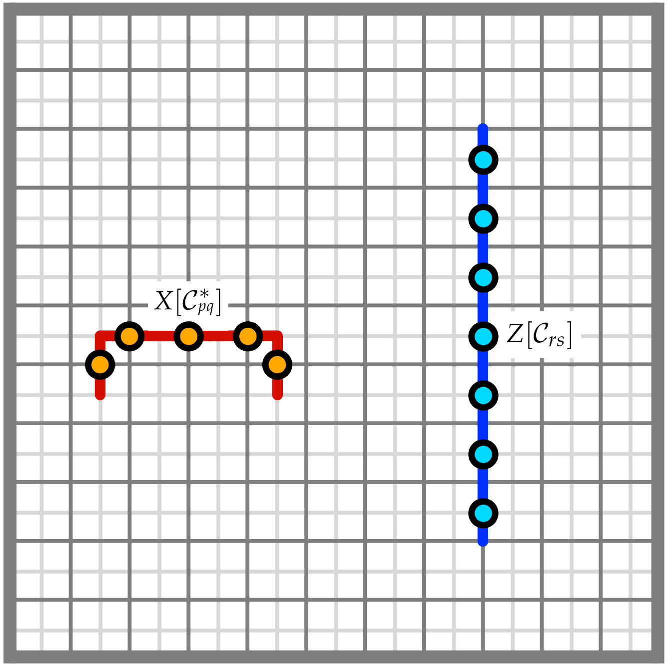
→ Here: $|v_1, v_2\rangle = |\,1\,,\,1\,\rangle$

2. Errors occur

→ Independent single-qubit errors:

$$\begin{aligned} \text{Spin-Flip}: &\quad \mathcal{PS} \ni |\Psi\rangle \longrightarrow \sigma_i^x\,|\Psi\rangle \notin \mathcal{PS} \\ \text{Phase-Flip}: &\quad \mathcal{PS} \ni |\Psi\rangle \longrightarrow \sigma_i^z\,|\Psi\rangle \notin \mathcal{PS} \end{aligned}$$

Error detection



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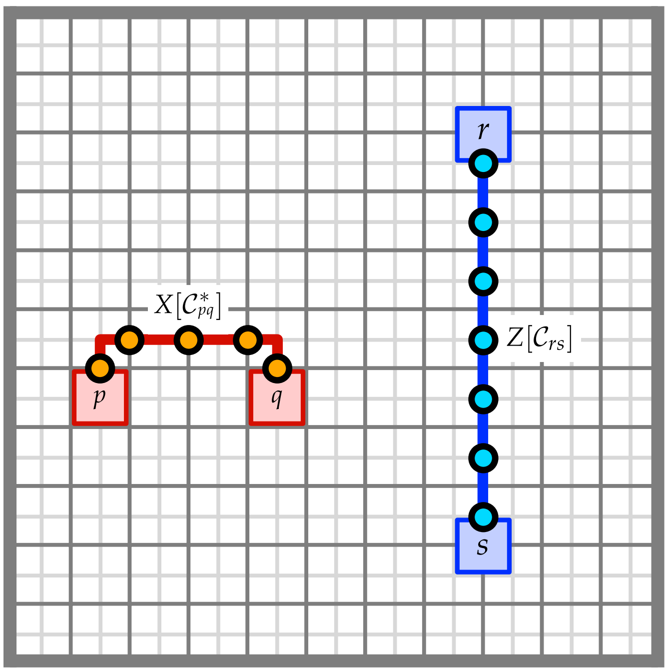
→ Errors = (open) String operators

Error detection



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→ Errors = (open) String operators

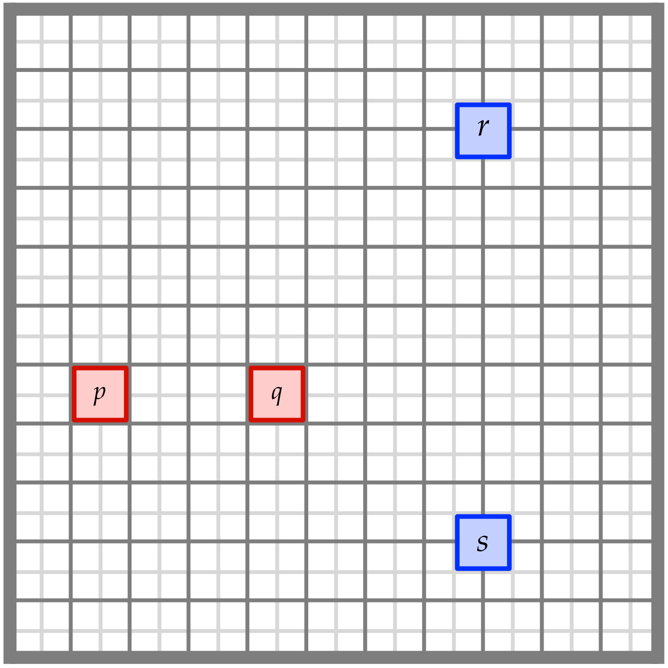
3. Syndrome measurement

→ Measure all star/plaquette operators:

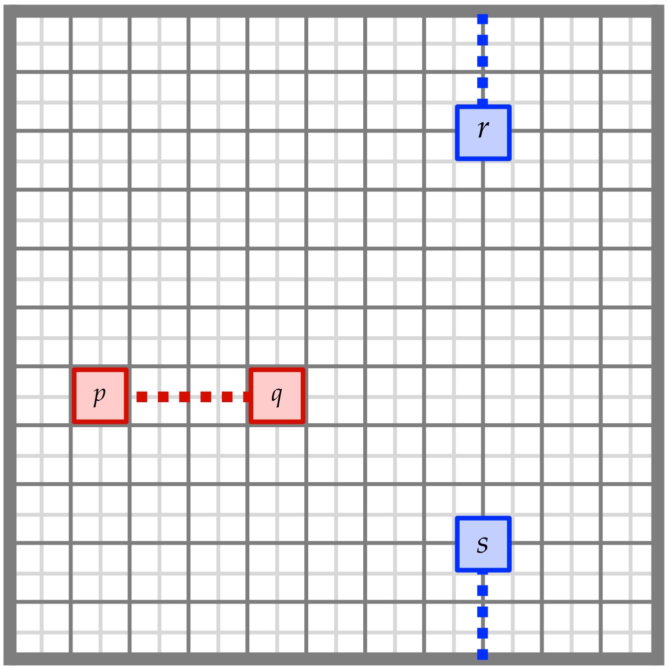
$$\begin{aligned} \text{Spin-Flip}: &\quad B_p \sigma_i^x\, |\Psi\rangle = -\sigma_i^x\, |\Psi\rangle \qquad i \in p \\ \text{Phase-Flip}: &\quad A_s \sigma_i^z\, |\Psi\rangle = -\sigma_i^z\, |\Psi\rangle \qquad i \in s \end{aligned}$$

→ Syndrome = Endpoints of error strings

Error correction



Error correction

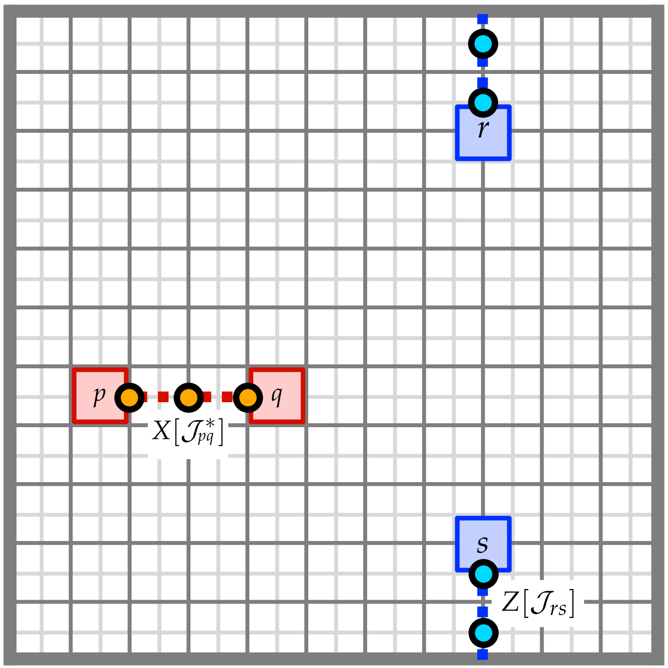


4. Computation of correction

→ Minimum weight perfect matching:

Pair & connect endpoints with shortest total path length

Error correction



4. Computation of correction

→ Minimum weight perfect matching:

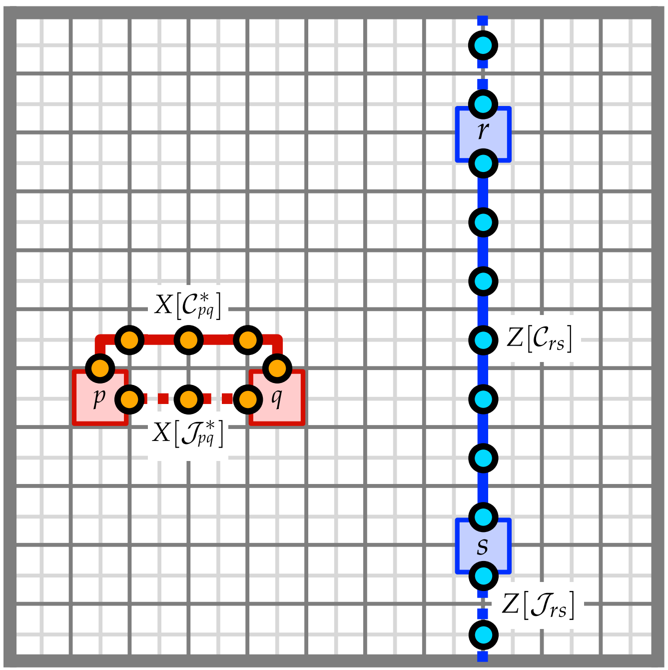
Pair & connect endpoints with shortest total path length

5. Application of correction

→ Error strings + correction operators:

$$X[\mathcal{C}^*_{pq}]X[\mathcal{J}^*_{pq}] = \mathbb{1} \qquad Z[\mathcal{C}_{rs}]Z[\mathcal{J}_{rs}] = Z_2$$

Error correction



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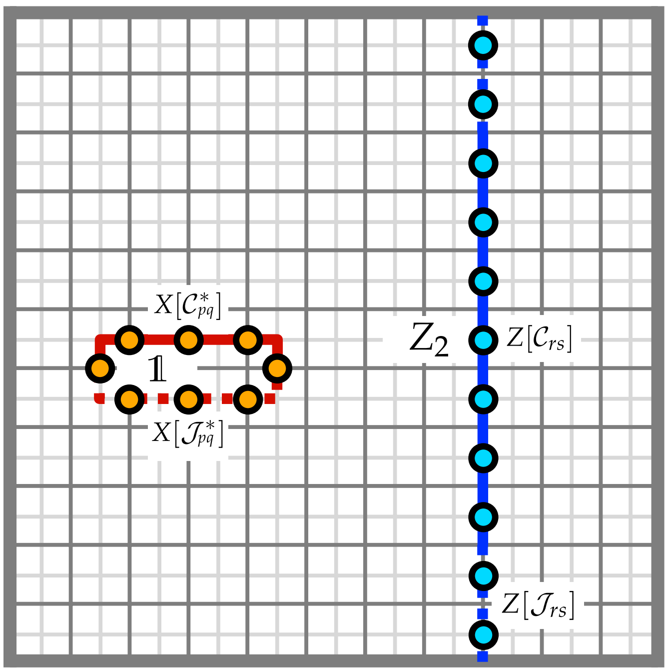

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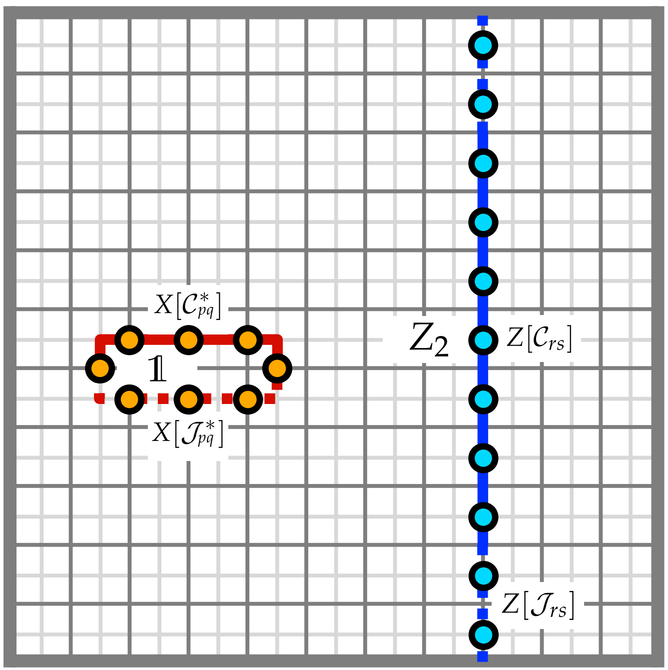
successful correction

failed correction (logical phase error)

Error correction



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failed correction (logical phase error)

Code distance

→ Correction without fail if less than

$$\left\lfloor \frac{L-1}{2} \right\rfloor$$ ← grows with L !

errors of the same type occured.

Experimental results

Revealing anyonic features in a toric code quantum simulation
→ J. K. Pachos *et. al.*, New J. Phys. 11, 083010 (2009)

Experimental demonstration of topological error correction
→ Xing-Can Yao *et. al.*, Nature 482, 489–494 (2012)

State preservation by repetitive error detection in a superconducting quantum circuit
→ J. Kelly *et. al.*, Nature 519, 66–69 (2015)

State based
(Photons, Josephson junctions)

Superconducting nanocircuits for topologically protected qubits
→ Sergey Gladchenko *et. al.*, Nature Physics 5, 48–53 (2009)

Hamiltonian based
(Josephson junctions)

→ NV-centers still missing …

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The End. Questions?