

# Topological Order with Fully-Symmetric Blockade Structures

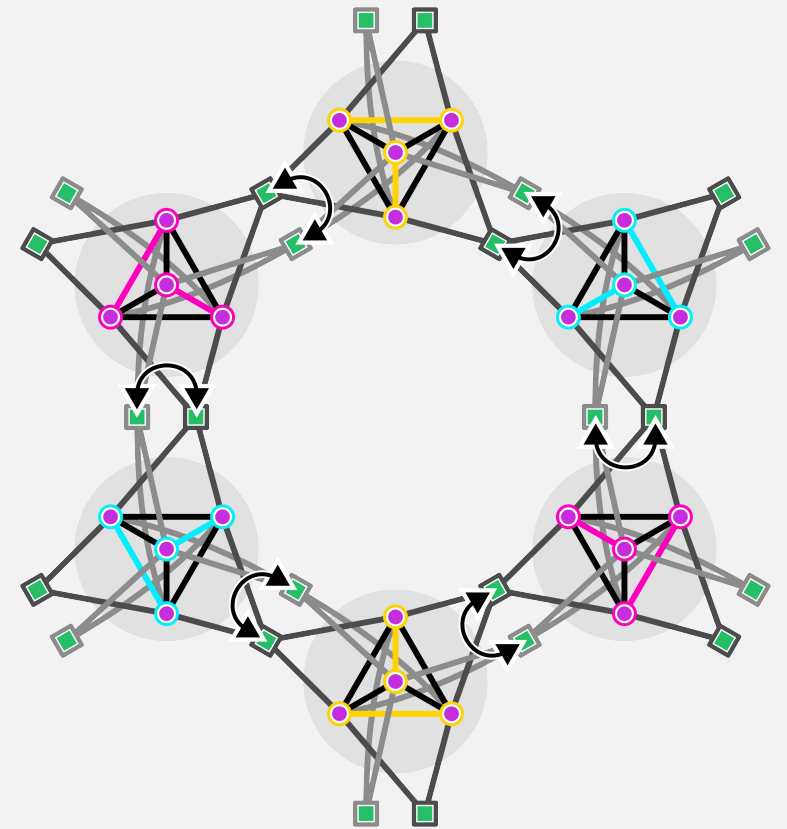
**Nicolai Lang**

**Institute for Theoretical Physics III**

Department of Physics, University of Stuttgart

**Journal Club @ Neupert Group, University of Zurich**

Zurich, December 2025



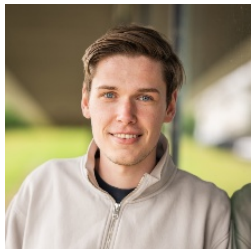
# People & Papers



Hans Peter Büchler



Tobias Maier

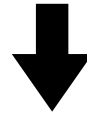


Simon Fell

## Functional completeness of planar Rydberg blockade structures

*Simon Stastny, Hans Peter Büchler, and NL*

Phys. Rev. B 108, 085138 (2023), doi:10.1103/PhysRevB.108.085138



## Topological order in symmetric blockade structures

*Tobias F. Maier, Hans Peter Büchler, and NL*

PRX Quantum 6, 030340 (2025), doi:10.1103/dtlf-2q82

Featured in  
Physics

This Talk



## Quantum doubles in symmetric blockade structures

*Hans Peter Büchler, Tobias F. Maier, Simon Fell, and NL*

arXiv: 2511.04414 (2025)

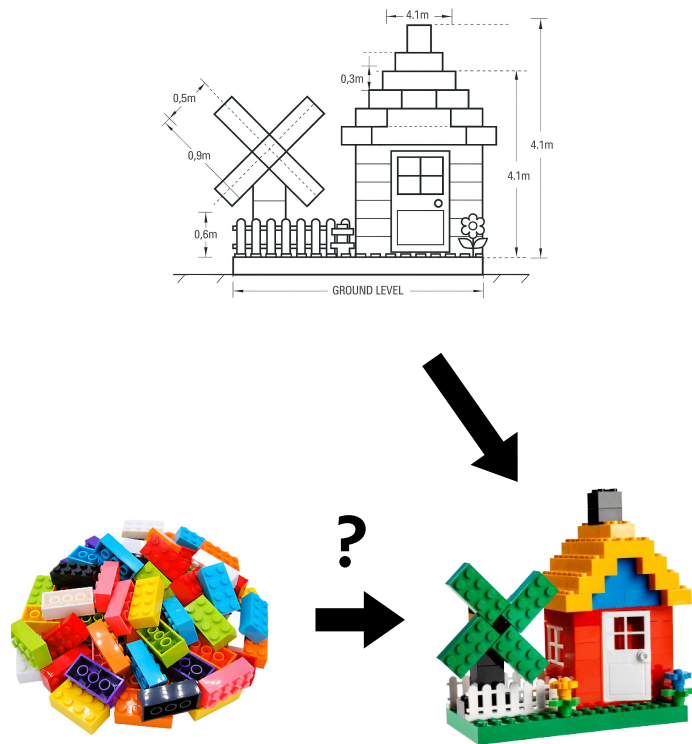
## Spectral gap of a blockade structure with $Z_2$ topological order

*Simon Fell, Tobias F. Maier, Hans Peter Büchler, and NL*

In preparation (2025)

# Rationale

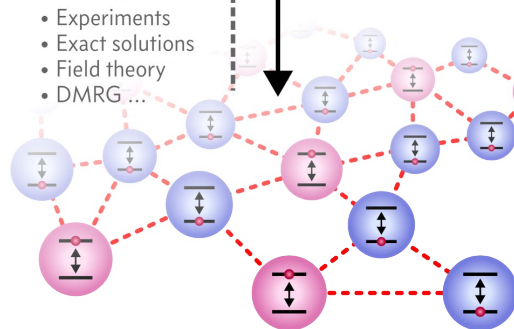
The „Inverse Problem“ of quantum many-body physics



**Emergence**

- Experiments
- Exact solutions
- Field theory
- DMRG ...

**Construction?**



**Goal**

- Complex correlations
- Long-range entanglement
- Robust

**Toolbox**

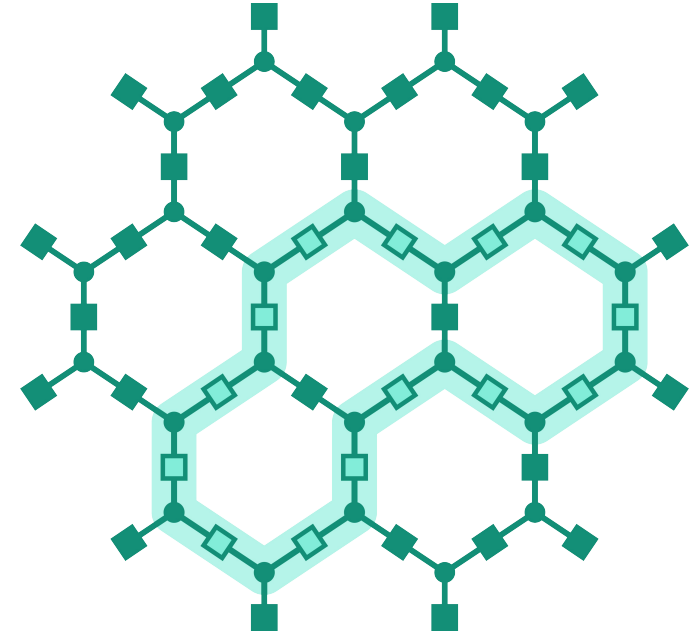
- Two-level systems
- Simple interactions
- Easy to control
- Potentially noisy

**Topological Order** <sup>1</sup> with

**Fully-Symmetric** <sup>3</sup>

**Blockade Structures** <sup>2</sup>

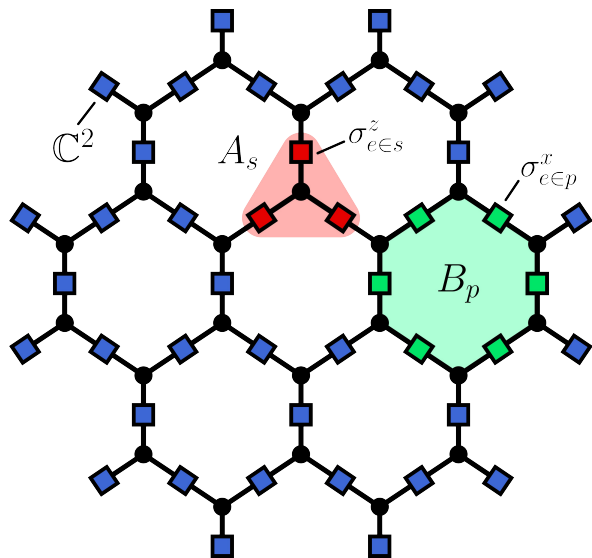
# Topological Order with Fully-Symmetric Blockade Structures



Topological Order and the Toric Code

# The Toric Code

Honeycomb lattice & 1 × Qubit per edge:



Ground state?

$$[A_s, B_p] = 0$$

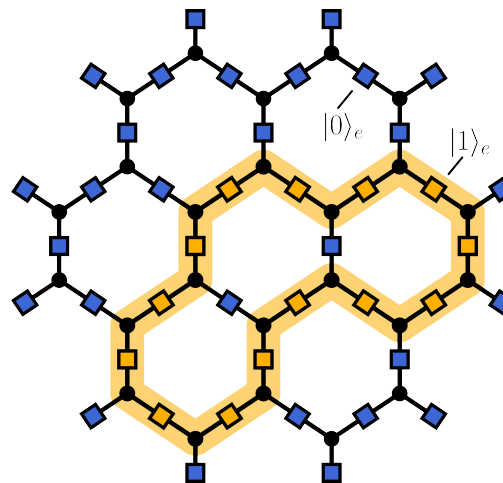
$$A_s^2 = \mathbb{1} \quad B_p^2 = \mathbb{1}$$

(exactly solvable)

$$A_s = +1 \quad \checkmark$$



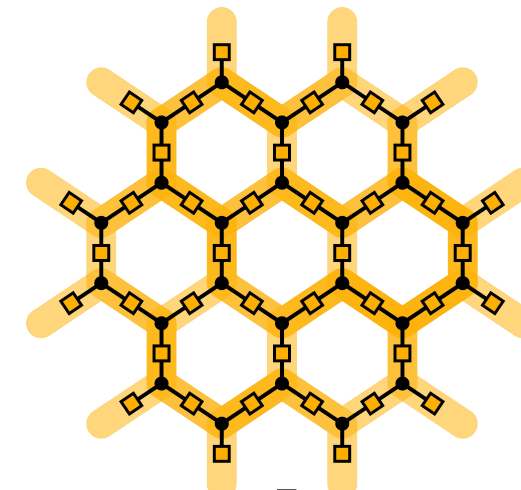
Loop constraint:



$$B_p = +1 \quad \checkmark$$



Loop condensation:



Ground state

$$|\Omega_{\text{TC}}\rangle \propto \sum_{n \in L_{\text{Loop}}} |n\rangle$$

All loop patterns

„Loop condensate“



Star operators

$$A_s = \prod_{e \in s} \sigma_e^z$$

Plaquette operators

$$B_p = \prod_{e \in p} \sigma_e^x$$

$$H_{\text{TC}} = -J_A \sum_{\text{Vertices } s} A_s - J_B \sum_{\text{Faces } p} B_p$$

Couplings  $J_A, J_B > 0$

Fault-tolerant quantum computation by anyons

A. Kitaev

Annals of Physics **303**(1), 2 (2003), doi:10.1016/s0003-4916(02)00018-0

String-net condensation: A physical mechanism for topological phases

M. A. Levin and X.-G. Wen

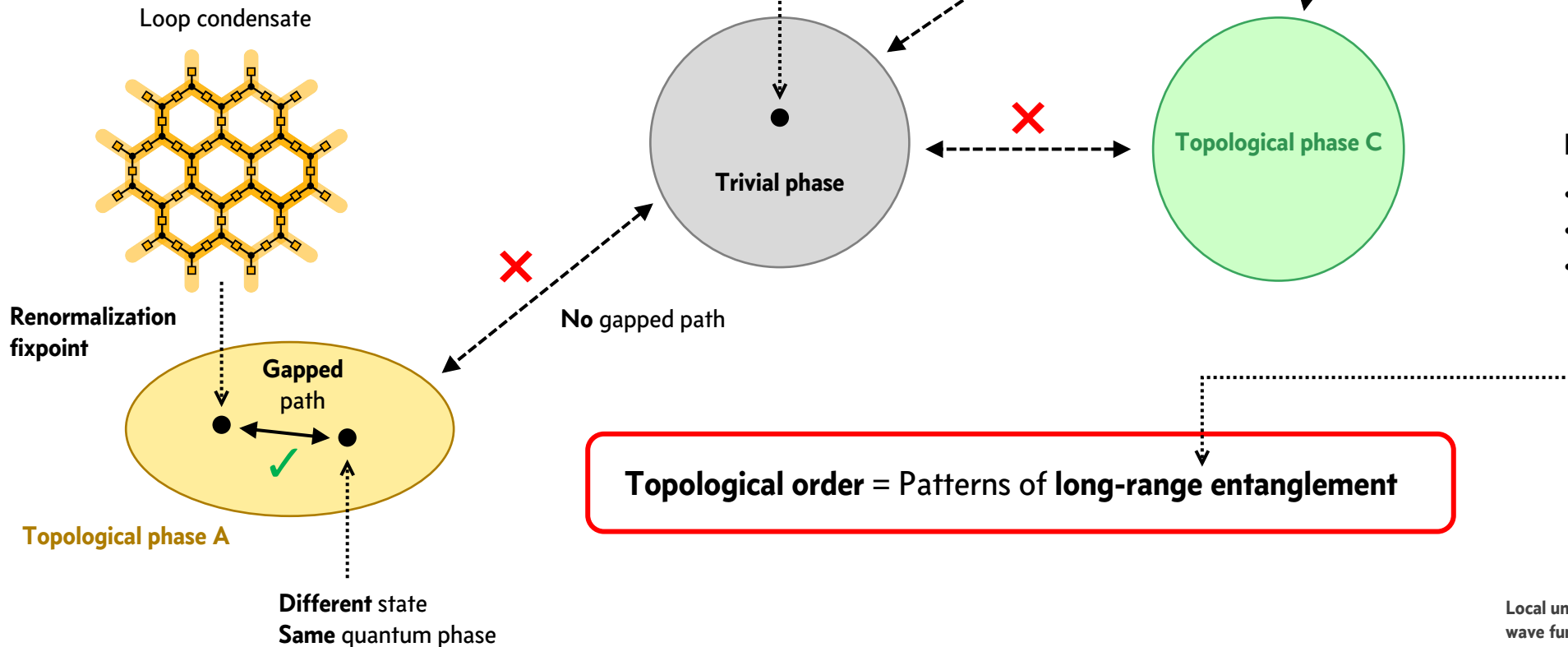
Physical Review B **71**, 045110 (2005), doi:10.1103/physrevb.71.045110

# Topological Order

True in 2D and higher ...  
(for bosonic systems)

Quantum phases **without** ...

- Spontaneous symmetry breaking
- Symmetry protection



**Fancy features:**

- **Anyonic** excitations
- Topological **ground state degeneracy**
- Described by **TQFTs**



Applications in  
**Quantum Computing ...**

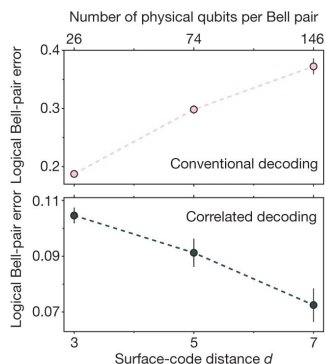
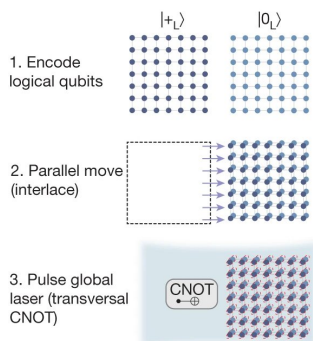
Local unitary transformation, long-range quantum entanglement,  
wave function renormalization, and topological order  
*Xie Chen, Zheng-Cheng Gu, and Xiao-Gang Wen*  
Phys. Rev. B **82**, 155138 (2010), doi:10.1103/PhysRevB.82.155138

# Applications

Topological quantum **computing** ✗  
 Topological quantum **memory** ✓

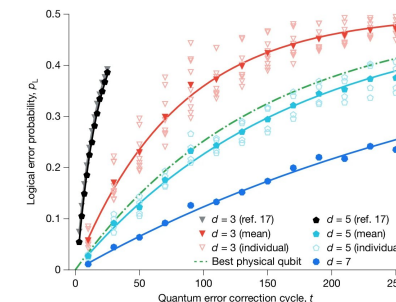
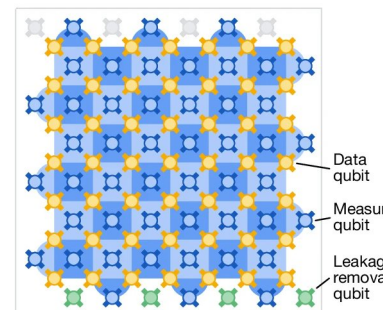
← Toric code = **Abelian** topological order

## Neutral atoms (Rydberg atoms)



Logical quantum processor based on reconfigurable atom arrays  
 Bluvstein et al.  
 Nature **626**, 58–65 (2024), doi:10.1038/s41586-023-06927-3

## Superconducting circuits (Transmons)



Quantum error correction below the surface code threshold  
 Google Quantum AI and Collaborators  
 Nature **638**, 920–926 (2025), doi:10.1038/s41586-024-08449-y

**Beware!**

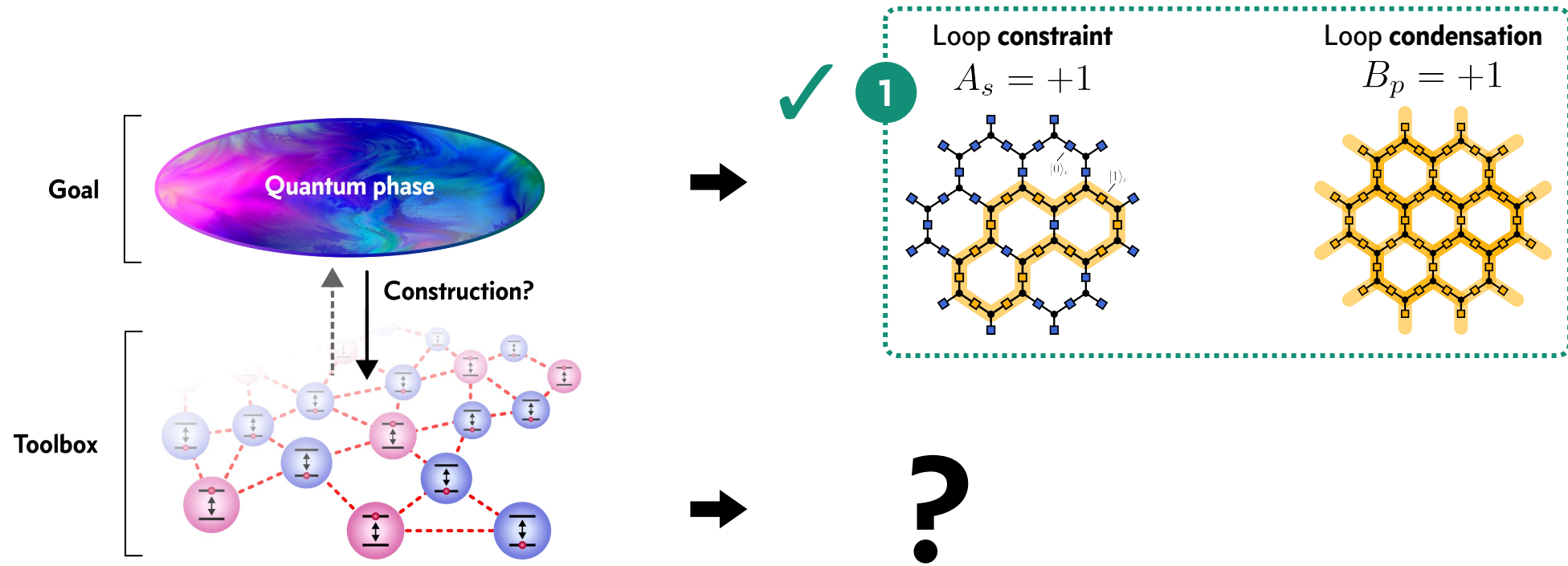
Loop condensate  $\neq$  Ground state of Hamiltonian



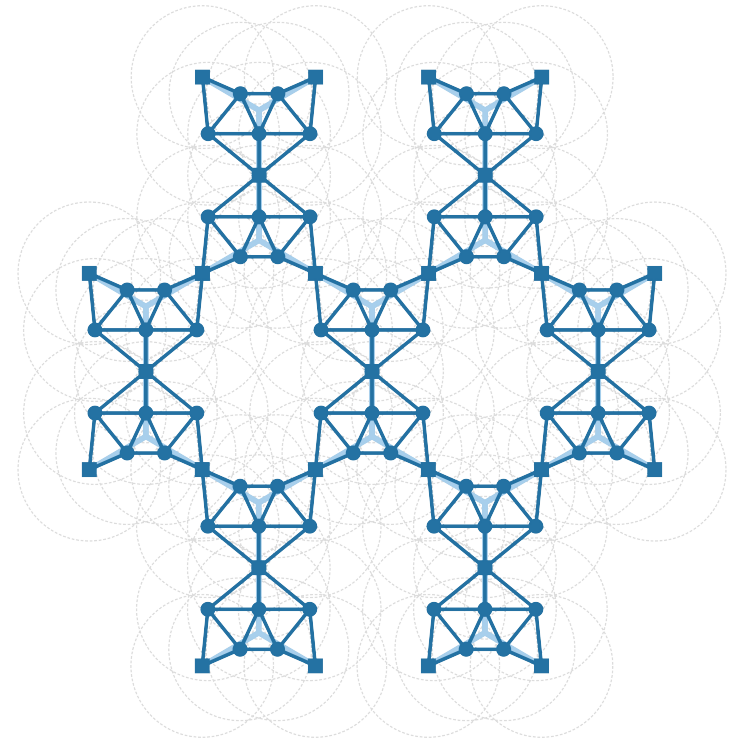
**This Talk:**

Design Hamiltonian  $\rightarrow$  Ground state = Loop condensate

# Toolbox?



# Topological Order with Fully-Symmetric Blockade Structures

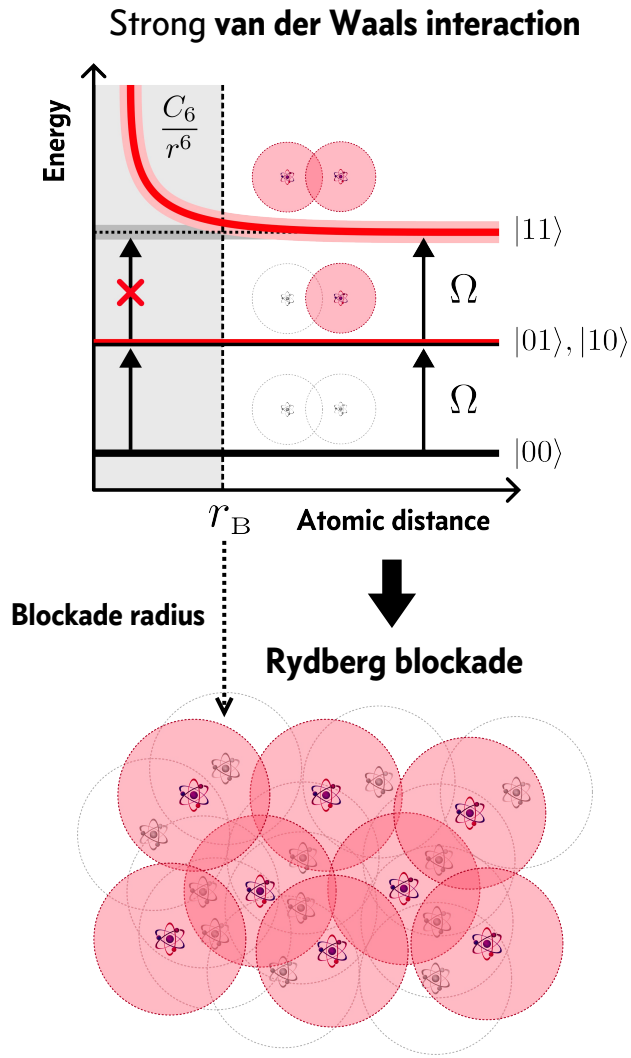


Functional Completeness of Blockade Structures

# Context & Motivation

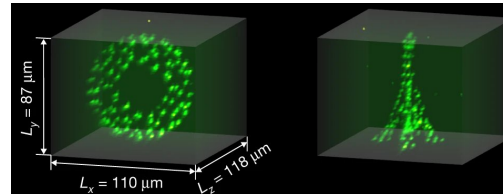
Versatile platform!

Many experiments ...

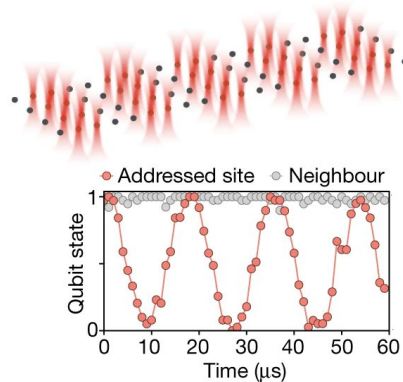


**Fast Quantum Gates for Neutral Atoms**  
*D. Jaksch, J. I. Cirac, P. Zoller, S. L. Rolston, R. Côté and M. D. Lukin*  
 Physical Review Letters **85**(10), 2208 (2000), doi:10.1103/physrevlett.85.2208

**Spatial control**  
 (Optical tweezers)



**Single-site addressability**

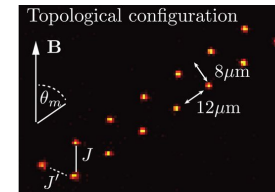


## Quantum simulation

**Probing many-body dynamics on a 51-atom quantum simulator**  
*Hannes Bernien et al.*  
 Nature **551**, 579–584 (2017), doi:10.1038/nature24622

**Observation of a symmetry-protected topological phase of interacting bosons with Rydberg atoms**  
*Sylvain de Léséleuc et al.*  
 Science **365** (6455), 775–780 (2019), doi:10.1126/science.aav9105

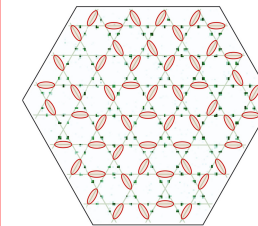
...



## Artificial quantum matter

**Prediction of Toric Code Topological Order from Rydberg Blockade**  
*Ruben Verresen, Mikhail D. Lukin, and Ashvin Vishwanath*  
 Phys. Rev. X **11**, 031005 (2021), doi:10.1103/PhysRevX.11.031005

**Probing topological spin liquids on a programmable quantum simulator**  
*G. Semeghini et al.*  
 Science **374** (6572), 1242–1247 (2021), doi:10.1126/science.abi8794

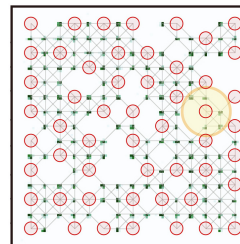


## Optimization („Geometric programming“)

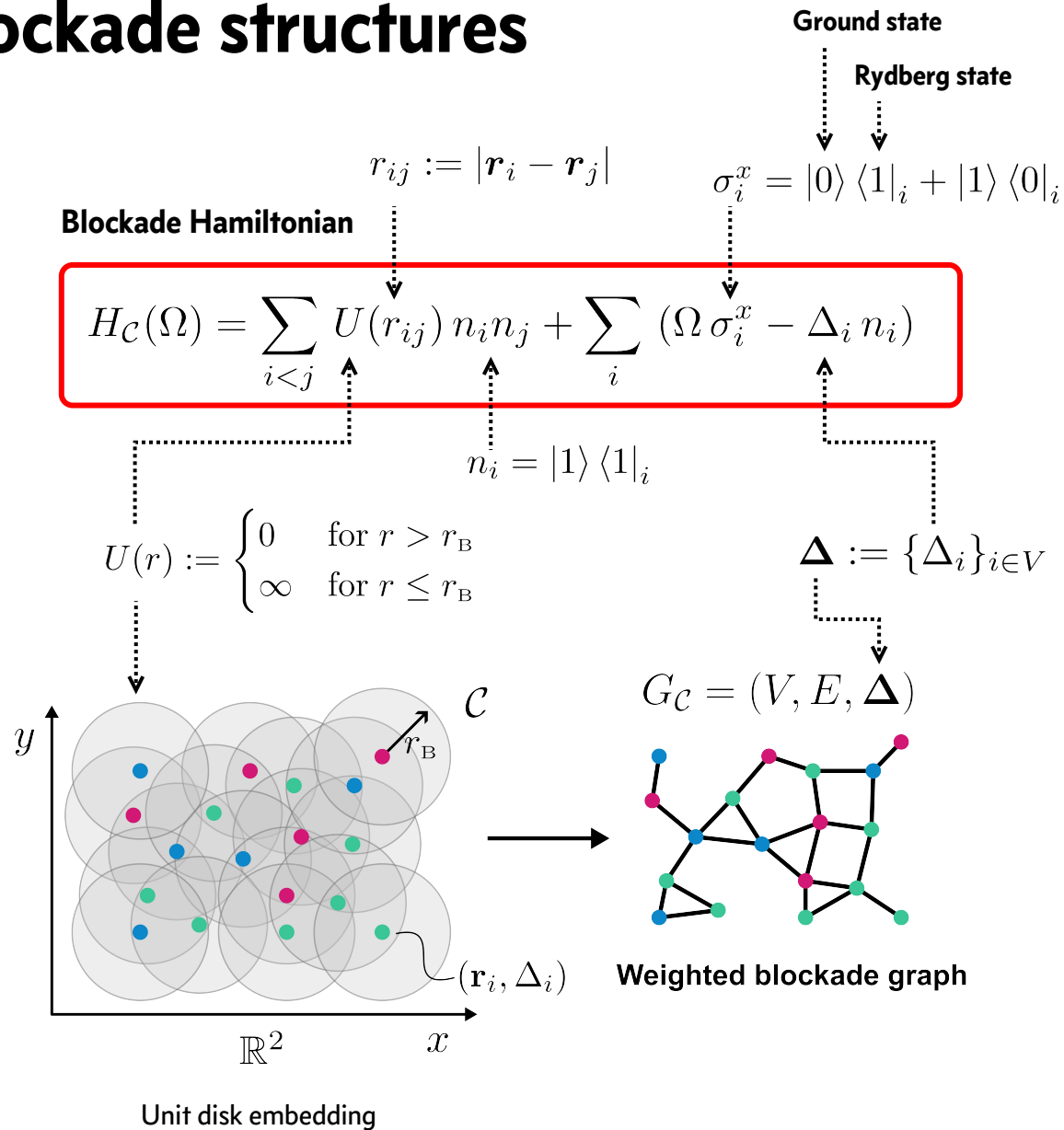
**Rydberg quantum wires for maximum independent set problems**  
*Minhyuk Kim et al.*  
 Nature Physics **18**, 755–759 (2022), doi:10.1038/s41567-022-01629-5

**Quantum optimization of maximum independent set using Rydberg atom arrays**  
*S. Ebadi et al.*  
 Science **376** (6598), 1209–1215 (2022), doi:10.1126/science.abo6587

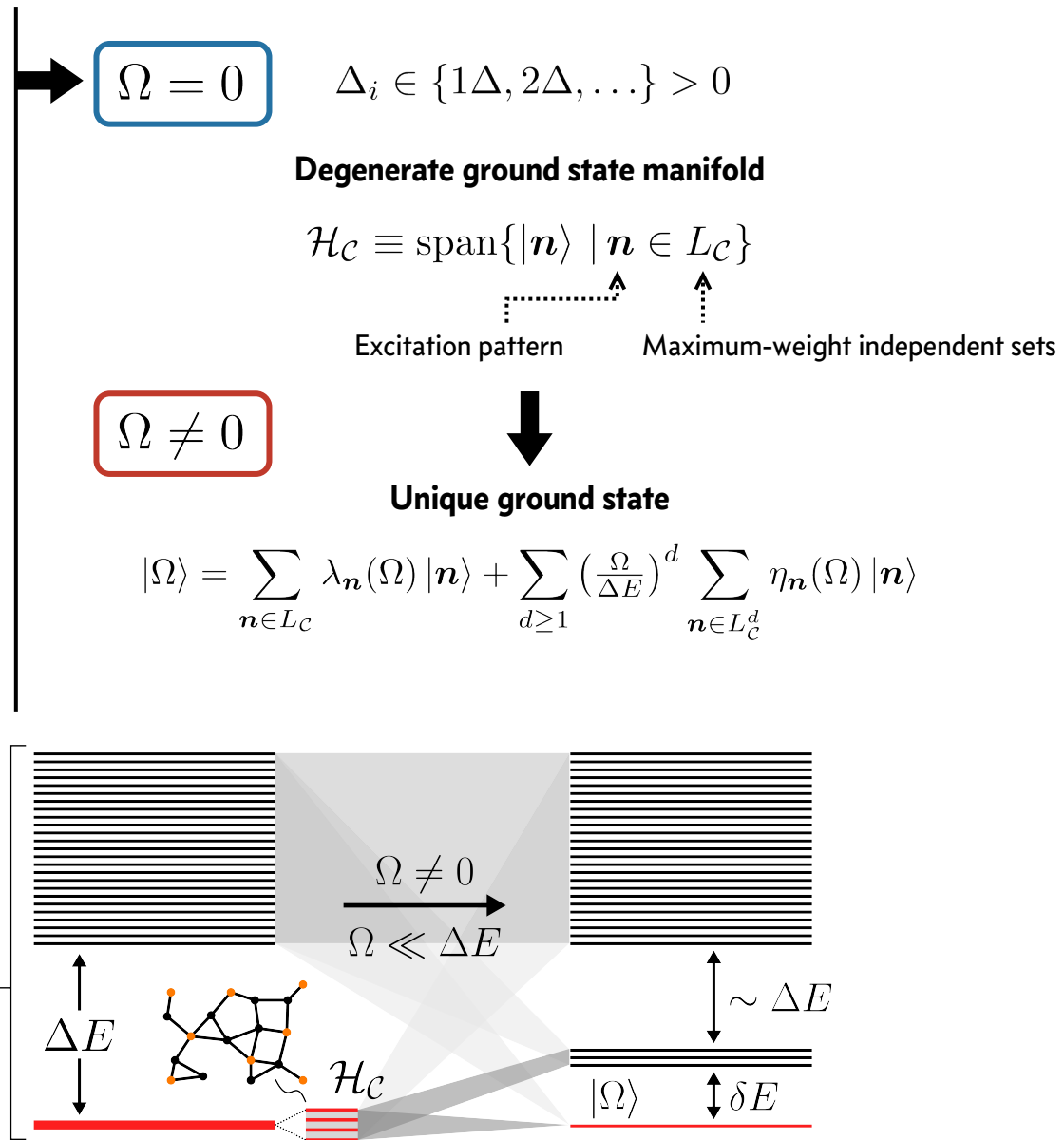
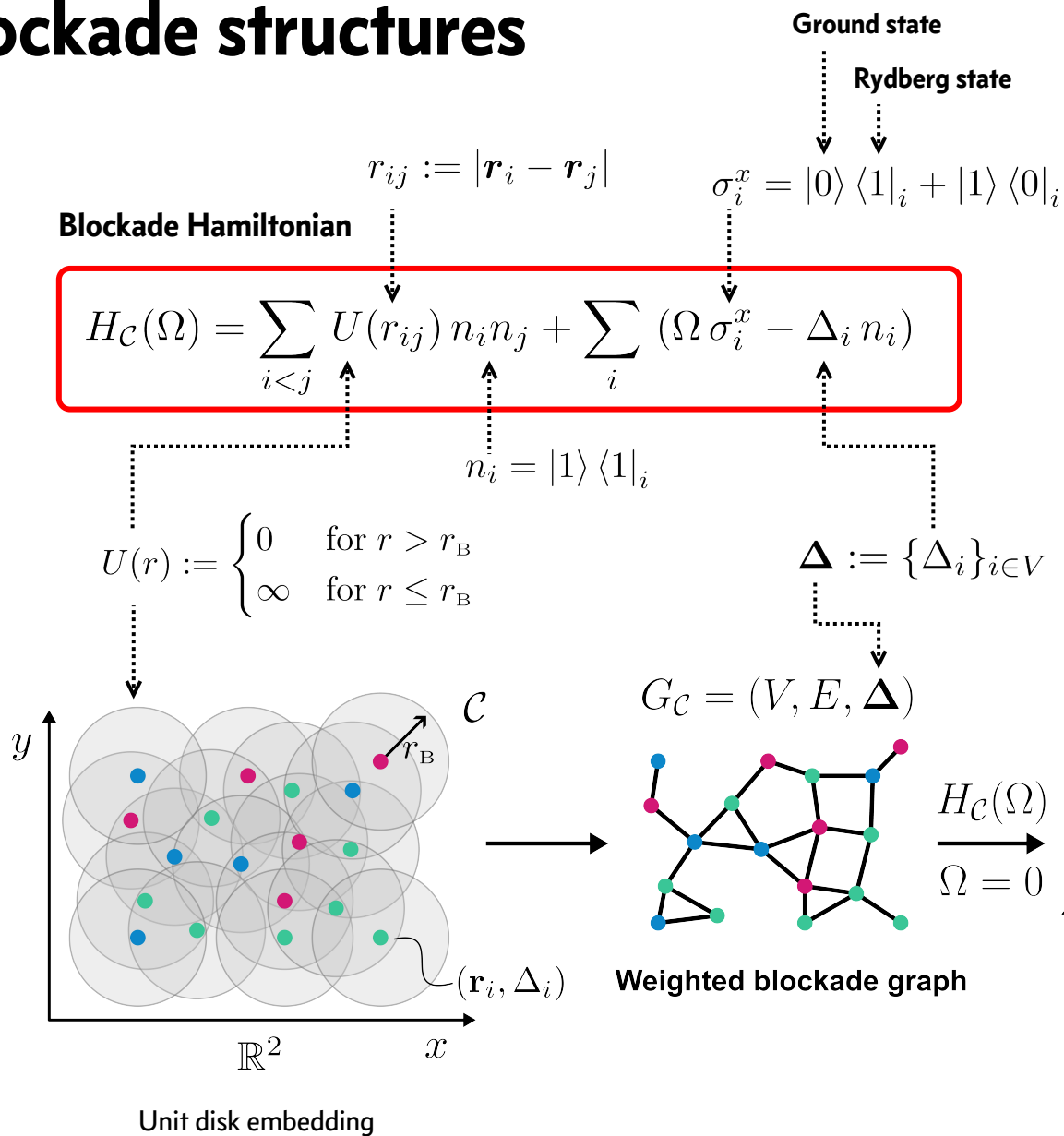
...



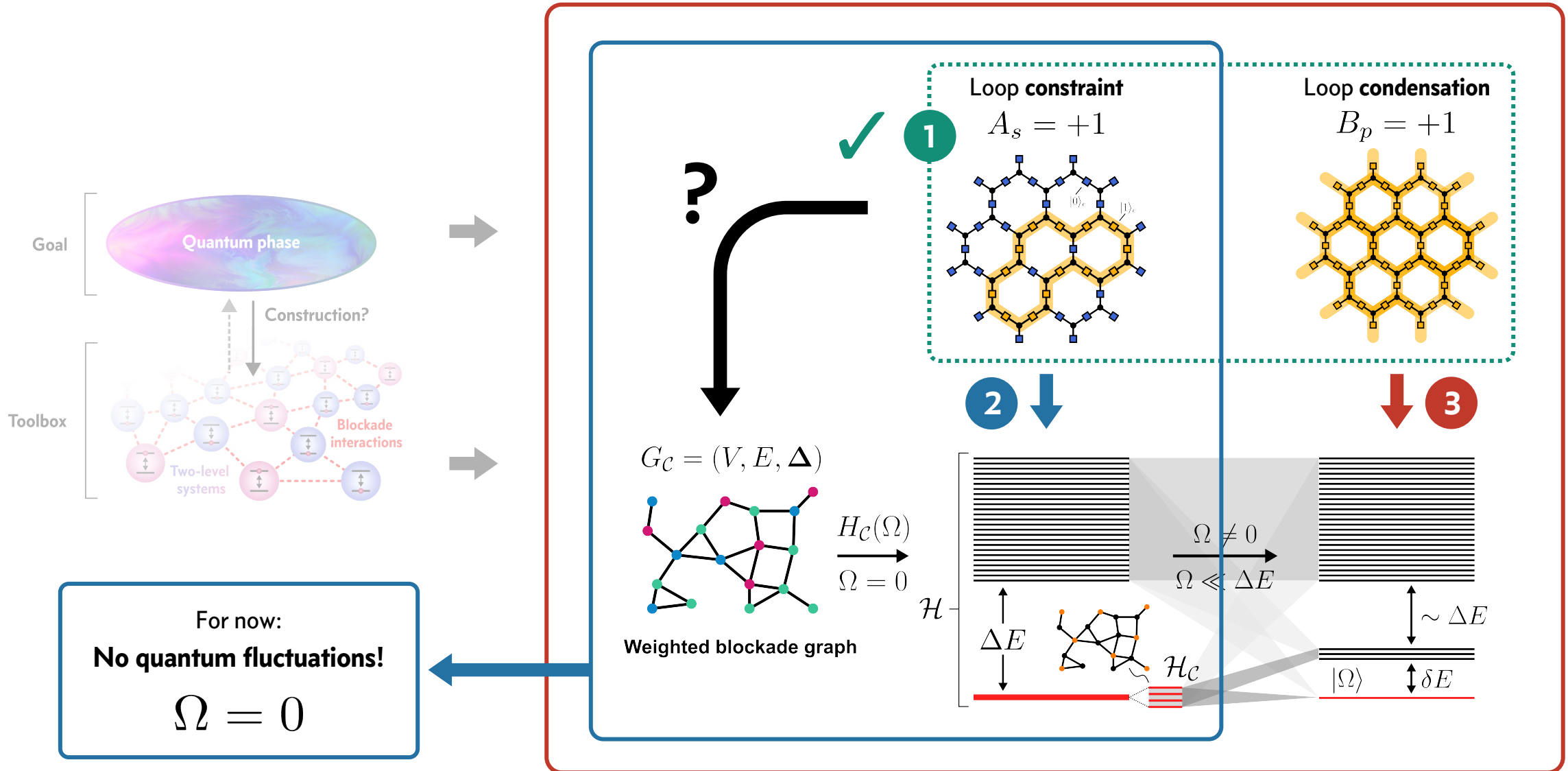
# Blockade structures



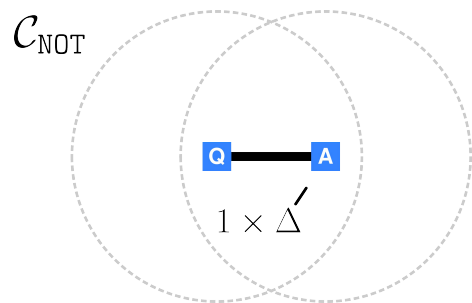
# Blockade structures



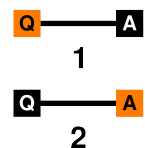
# Objective for the remainder of this talk ...



# Logic with Atoms



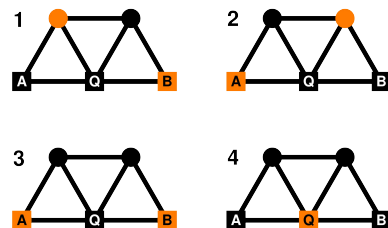
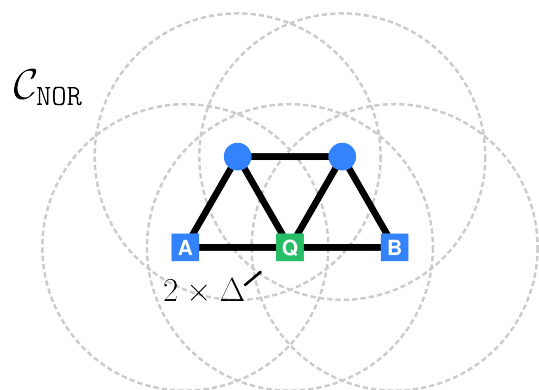
Unit disk embedding



Ground states

Atom	A	Q
Ground state 1	0	1
Ground state 2	1	0

Truth table



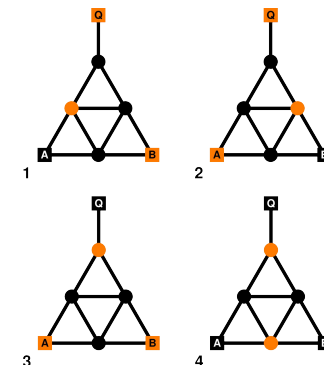
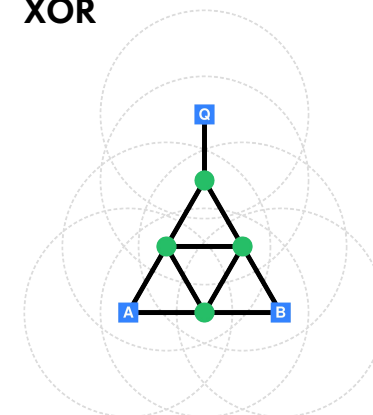
Atom	A	B	Q
Ground state 1	0	1	0
Ground state 2	1	0	0
Ground state 3	1	1	0
Ground state 4	0	0	1

$$\text{NOR}(x, y) \equiv x \downarrow y := \overline{x \vee y}$$

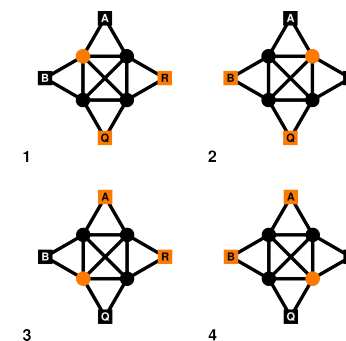
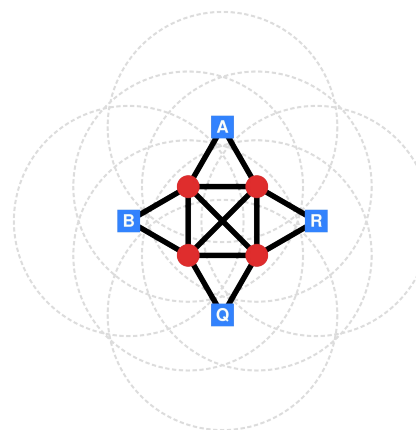
**Universal gate**

## Other gates ... (useful later)

### XOR

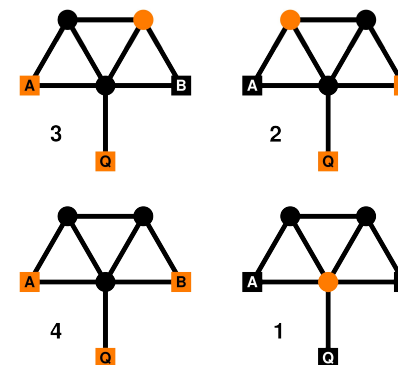
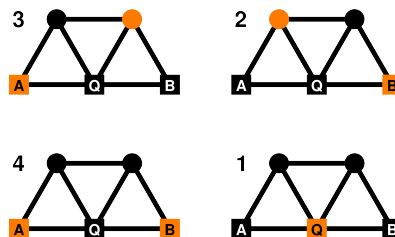
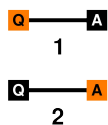
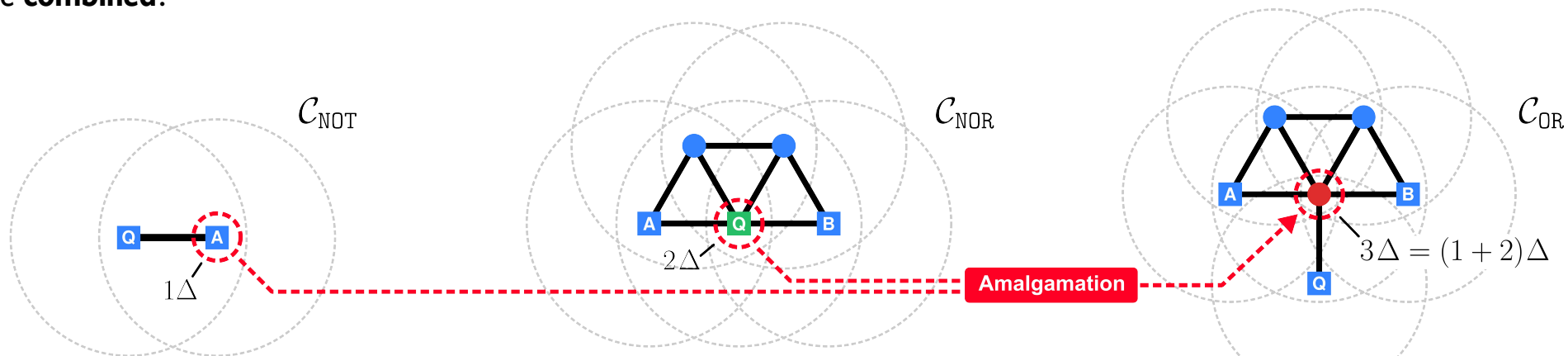


### Inverted crossing (ICRS)

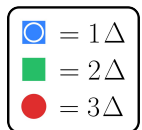


# Amalgamation

Can gates be combined?



Detunings



Atom	A	B	Q
Ground state 1	0	0	0
2	0	1	1
3	1	0	1
4	1	1	1

$$Q = \overline{A \downarrow B}$$

$$= A \vee B$$

➔ Gates can be combined to Boolean circuits!

# Functional Completeness

## NOR-Gate & Amalgamation

- Constructive (but convoluted)
- Unit-disk embeddable
- Ancillas needed

For every **Boolean function**  $f$  exists a **blockade structure**  $\mathcal{C}_f$  with **ground state manifold**

$$\mathcal{H}_{\mathcal{C}_f} = \text{span}\{ |\mathbf{n}\rangle \mid f(\mathbf{n}) = 1 \}$$

→ Blockade structures are **functionally complete**

E.g. **String-net Hilbert spaces:**

- $\mathbb{Z}_2$  Toric Code
- Fibonacci fusion rules
- ...

### Artificial quantum matter

→ ( Local constraints )

Functional completeness of planar Rydberg blockade structures

*Simon Stastny, Hans Peter Büchler, and NL*

Phys. Rev. B 108, 085138 (2023), doi:10.1103/PhysRevB.108.085138

### Optimization

( „Geometric programming“ )

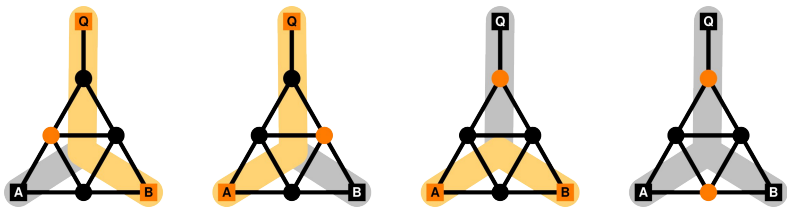
Quantum optimization with arbitrary connectivity using Rydberg atom arrays

*M.-T. Nguyen et al.*

PRX Quantum 4, 010316 (2023), doi:10.1103/PRXQuantum.4.010316

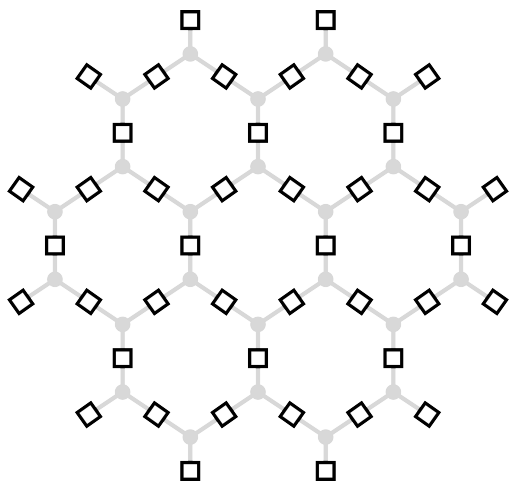
# Example: Constructing a Loop Hilbert Space

Observation

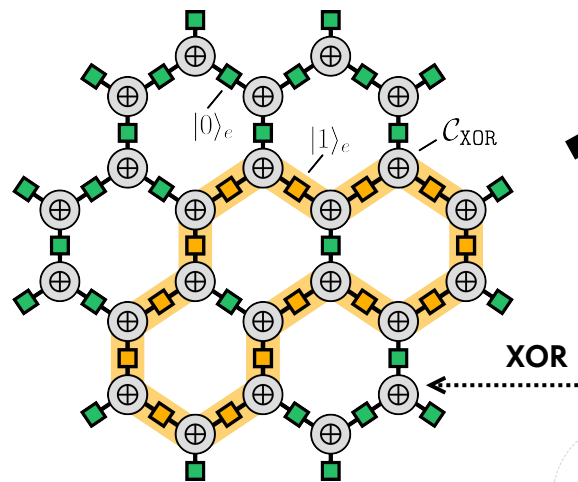


Trivalent **loop** constraint = **XOR** constraint

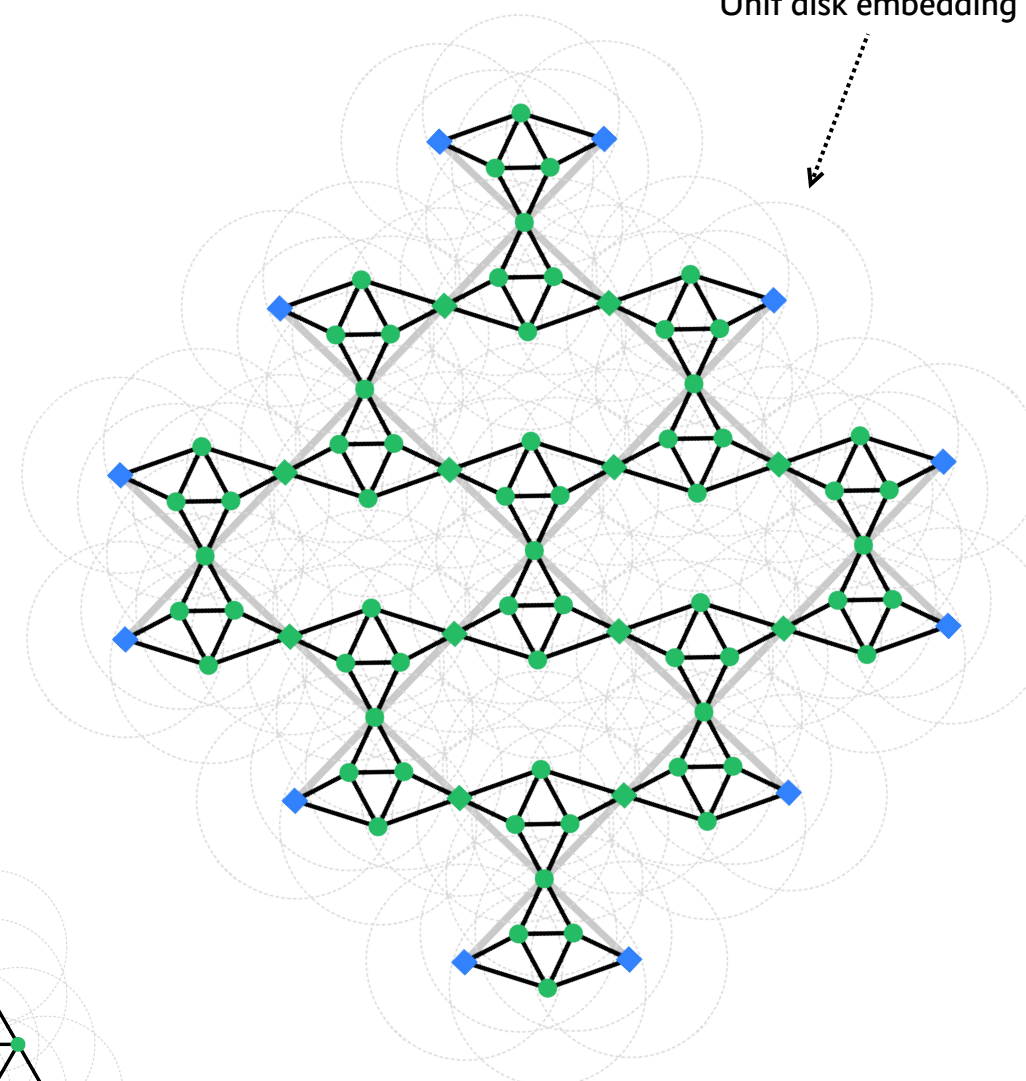
1 × Atom per edge:



Amalgamate **XOR** gates on sites:

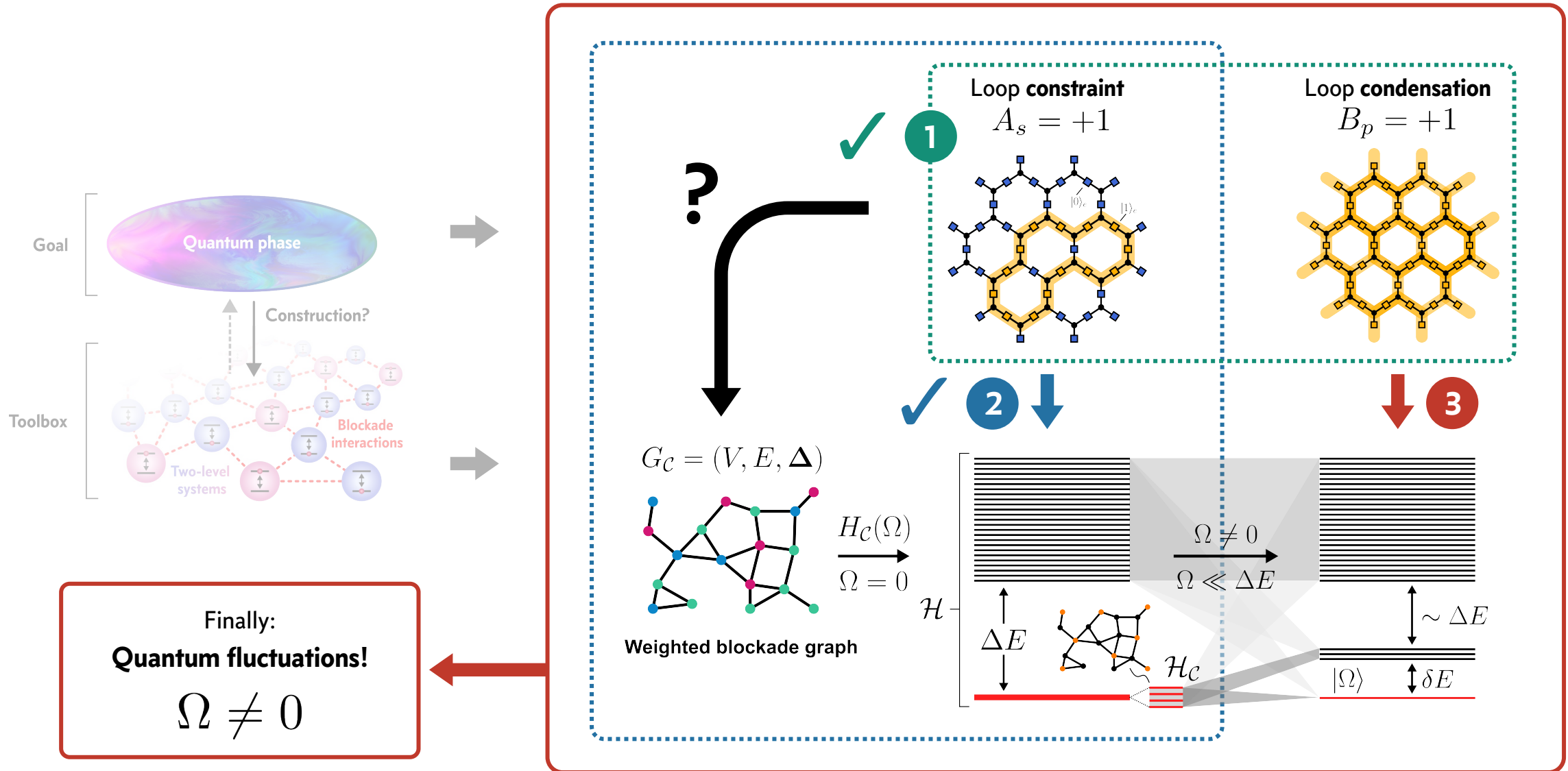


Unit disk embedding

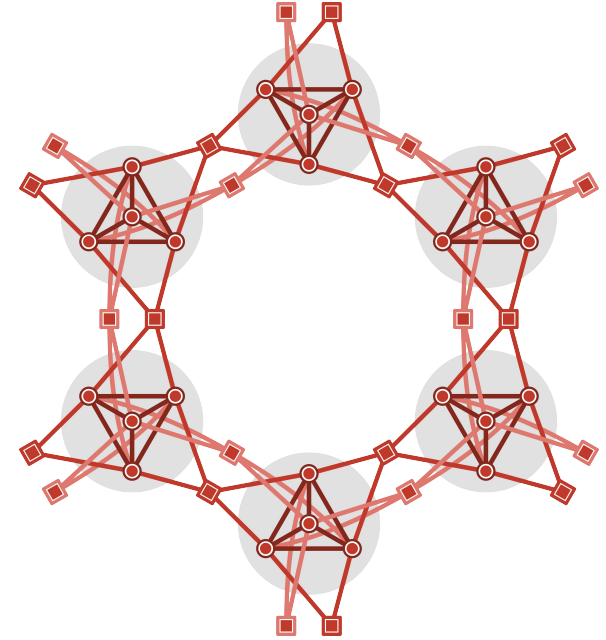


(Honeycomb lattice ~ Square lattice)

# Objective for the remainder of this talk ...



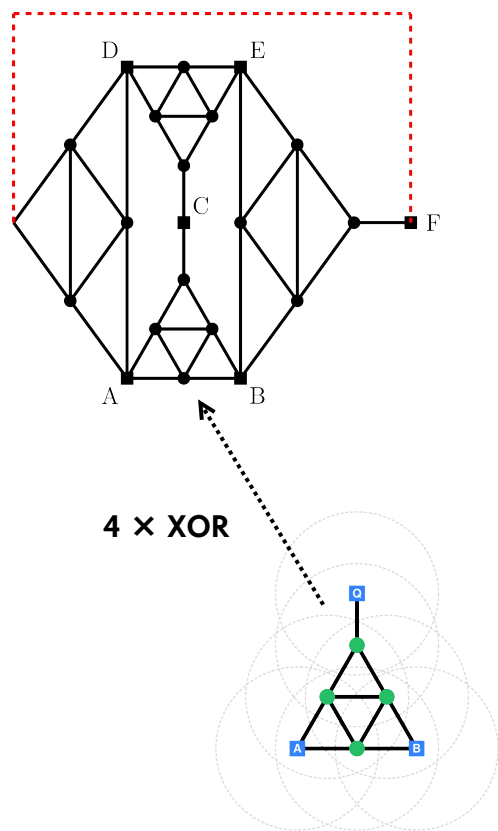
# Topological Order with **Fully-Symmetric** Blockade Structures



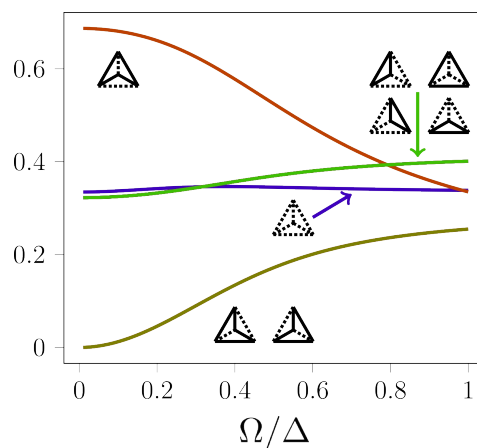
Full Symmetry and Equal-weight Superpositions

# First try: Toric Code + Quantum Fluctuations

Simplification:  
Toric code on minimal sphere (Tetrahedron)



Exact diagonalization



Explanation:  
Hamming distances

	$ \Delta\rangle$	$ \Delta\rangle$	$ \Delta\rangle$	$ \Delta\rangle$	$ \Delta\rangle$	$ \Delta\rangle$	$ \Delta\rangle$	$ \Delta\rangle$
$ \Delta\rangle$	-	10	14	10	10	8	10	14
$ \Delta\rangle$		-	10	16	10	8	16	10
$ \Delta\rangle$			-	10	14	8	10	10
$ \Delta\rangle$				-	10	12	12	10
$ \Delta\rangle$					-	8	10	14
$ \Delta\rangle$						-	12	8
$ \Delta\rangle$							-	10
$ \Delta\rangle$								-

Some transitions dominate others ...

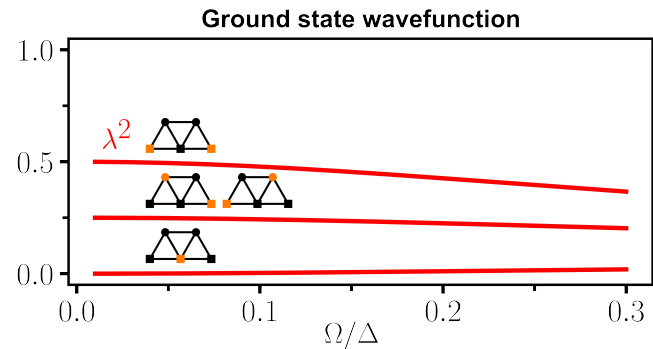
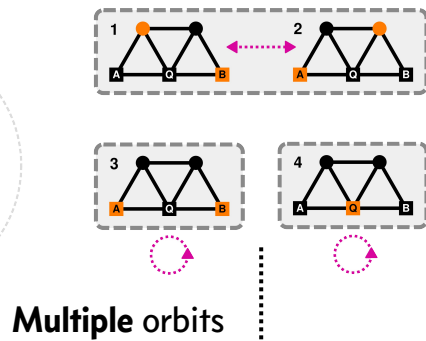
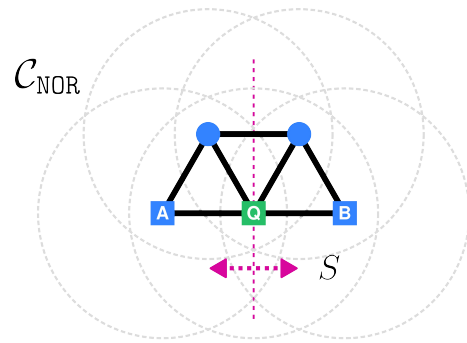


Not equal-weight!

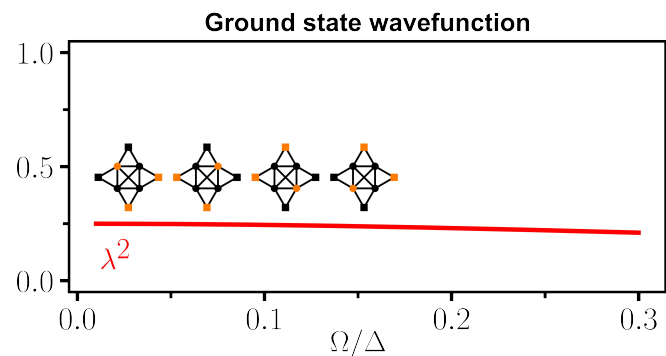
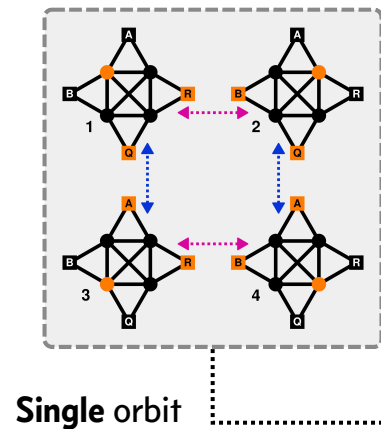
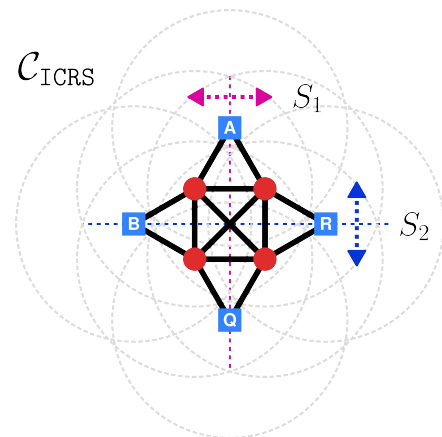
How to enforce **equal-weight** superpositions?

Quantum many-body phases with gauge constraints  
Tobias F. Maier  
Bachelor Thesis, University of Stuttgart (2023)

# Second try: Logic Gates + Quantum Fluctuations



Not equal-weight ...  
?



Equal-weight!  
?

ICRS = Inverted Crossing

# Full Symmetry

Definition

$$\phi : V \rightarrow V \quad G_C = (V, E, \Delta)$$

Graph automorphism iff:

$$\{\phi(i), \phi(j)\} \in E \Leftrightarrow \{i, j\} \in E$$

$$\& \Delta_i = \Delta_{\phi(i)}$$

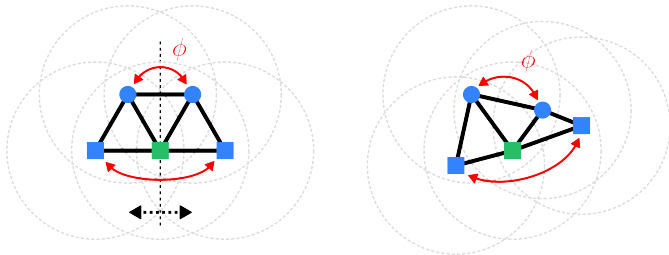


Automorphism group

$$\mathcal{A}_C := \text{Aut}(G_C)$$

Beware!

Graph automorphisms  $\neq$  Euclidean symmetries:



# Full Symmetry

## Definition

$$\phi : V \rightarrow V \quad G_C = (V, E, \Delta)$$

Graph automorphism iff:

$$\{\phi(i), \phi(j)\} \in E \Leftrightarrow \{i, j\} \in E$$

$$\& \Delta_i = \Delta_{\phi(i)}$$



Automorphism group

$$\mathcal{A}_C := \text{Aut}(G_C)$$

## Observation

Unitary action:  $U_\phi |n\rangle := |\phi \cdot n\rangle$

Graph automorphisms  $\rightarrow$  Symmetries:

$$\phi \in \mathcal{A}_C \Rightarrow [H_C, U_\phi] = 0$$

## Definition

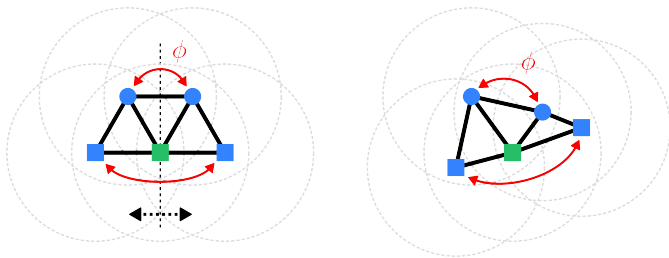
$\mathcal{C}$  fully symmetric iff:

$$\underbrace{\frac{|\mathcal{L}_C|}{|\mathcal{A}_C|}}_{\# \text{ Orbits}} = 1$$

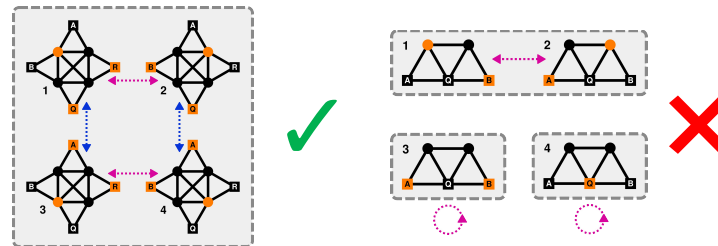
All ground state patterns      All automorphisms

**Beware!**

Graph automorphisms  $\neq$  Euclidean symmetries:



Examples:



# Full Symmetry

## Definition

$$\phi : V \rightarrow V \quad G_C = (V, E, \Delta)$$

Graph automorphism iff:

$$\{\phi(i), \phi(j)\} \in E \Leftrightarrow \{i, j\} \in E$$

$$\& \quad \Delta_i = \Delta_{\phi(i)}$$

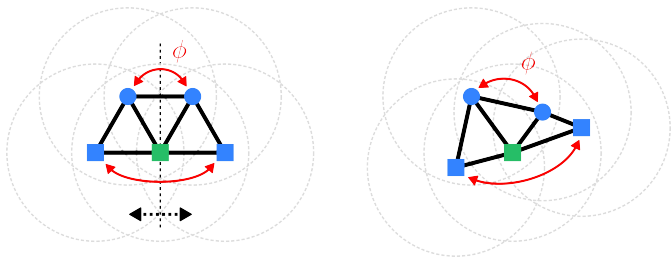


Automorphism group

$$\mathcal{A}_C := \text{Aut}(G_C)$$

**Beware!**

Graph automorphisms  $\neq$  Euclidean symmetries:



## Observation

Unitary action:  $U_\phi |n\rangle := |\phi \cdot n\rangle$

Graph automorphisms  $\rightarrow$  Symmetries:

$$\phi \in \mathcal{A}_C \Rightarrow [H_C, U_\phi] = 0$$

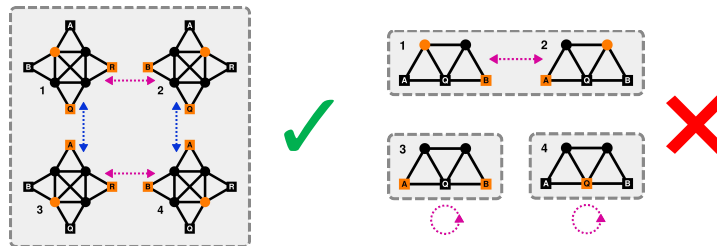
## Definition

$\mathcal{C}$  fully symmetric iff:

$$\underbrace{|\mathcal{L}_C / \mathcal{A}_C|}_{\# \text{ Orbits}} = 1$$

All ground state patterns All automorphisms

Examples:



## Theorem

$\mathcal{C}$  finite & fully symmetric



$$H_C(\Omega \neq 0)$$

has unique ground state

$$|\Omega\rangle = \lambda(\Omega) \sum_{n \in \mathcal{L}_C} |n\rangle + \sum_{d \geq 1} \left(\frac{\Omega}{\Delta E}\right)^d \sum_{n \in \mathcal{L}_C^d} \eta_n(\Omega) |n\rangle$$

In particular:

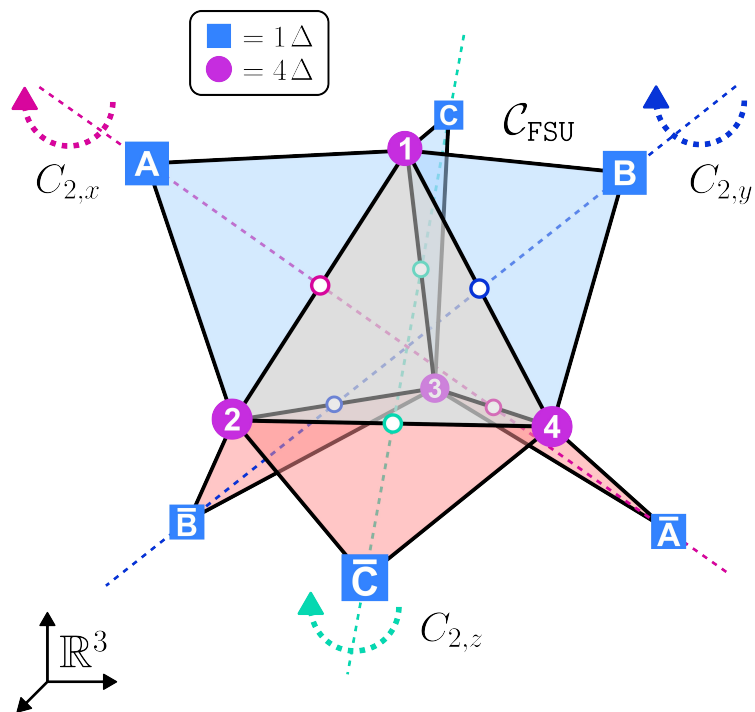
$$U_\phi |\Omega\rangle = |\Omega\rangle$$



We want:

**Loop constraint & Full symmetry**

# Preparation: The Fully-symmetric Universal Gate

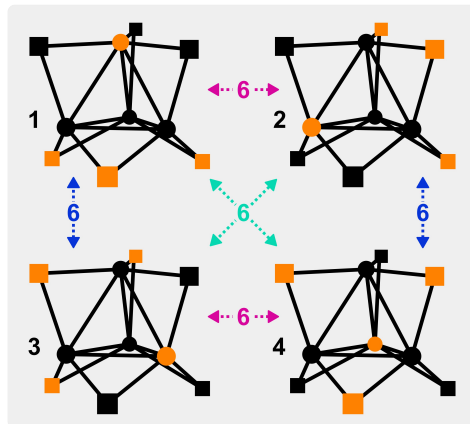


Automorphism group:

$$\mathcal{A}_{\text{FSU}} \simeq S_4 \supset V = \{\mathbb{1}, C_{2,x}, C_{2,y}, C_{2,z}\}$$

Tetrahedral symmetry

Klein four-group



Gate	A (Input)		B (Input)		XOR	XNOR	NOR	INH <sub>AB</sub>	AND	INH <sub>BA</sub>
Atom	A	$\bar{A}$	B	$\bar{B}$	C	$\bar{C}$	1	2	3	4
Ground state 1	0	1	0	1	0	1	1	0	0	0
Ground state 2	0	1	1	0	1	0	0	1	0	0
Ground state 3	1	0	0	1	1	0	0	0	0	1
Ground state 4	1	0	1	0	0	1	0	0	1	0

XOR

XNOR

$$A \oplus B = C \Leftrightarrow \overline{A \oplus B} = \bar{C} \Leftrightarrow \overline{A \odot B} = \bar{C}$$

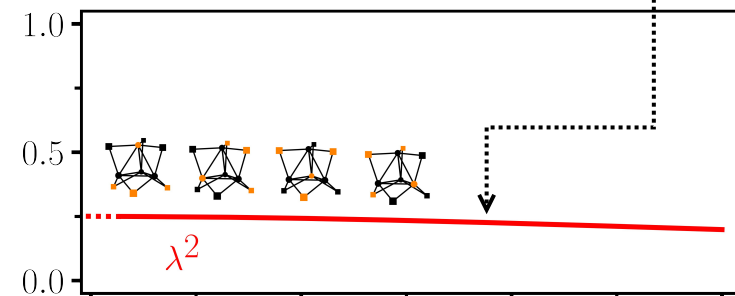
Upper "wings"

Lower "wings"

Single orbit

Fully symmetric

Ground state wavefunction



# Constructing a Fully-symmetric Loop Hilbert Space

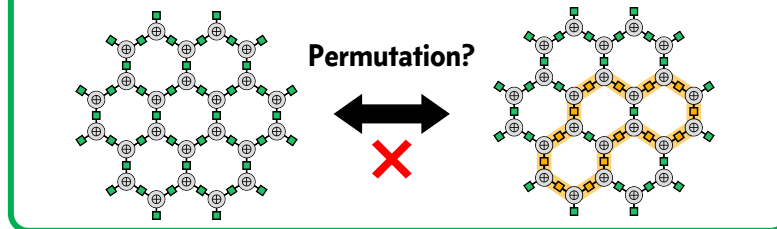
## Observation 1

$$\underbrace{|Lc / \mathcal{A}c|}_{\# \text{Orbits}} \stackrel{\text{FS}}{=} 1 \quad \xRightarrow{\text{Burnside}} \quad |\mathcal{A}c| \geq |Lc|$$

Extensive!  
↓  
Local automorphisms needed!

&

## Observation 2



Encode qubits on edges  
in two atoms  
that are in blockade!

# Constructing a Fully-symmetric Loop Hilbert Space

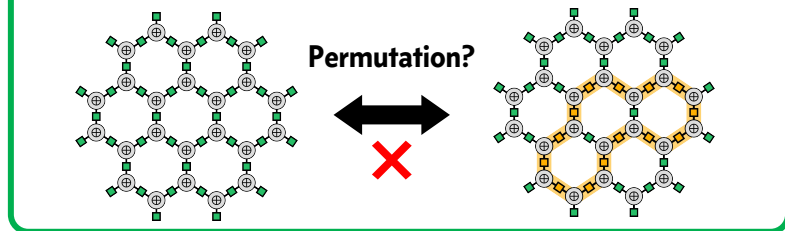
## Observation 1

$$\underbrace{|Lc / \mathcal{A}c|}_{\# \text{Orbits}} \stackrel{\text{Burnside}}{\stackrel{\text{FS}}{=} 1} \Rightarrow \text{Extensive!} \downarrow |\mathcal{A}c| \geq |Lc|$$

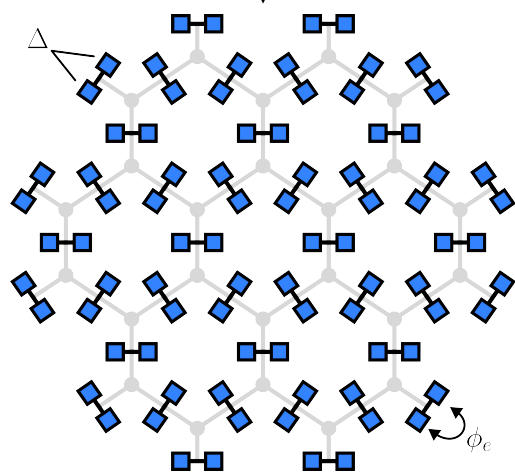
Local automorphisms needed!

&

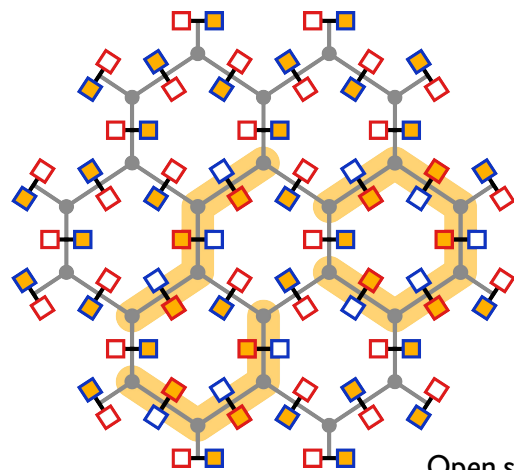
## Observation 2



Encode qubits on edges in two atoms that are in blockade!

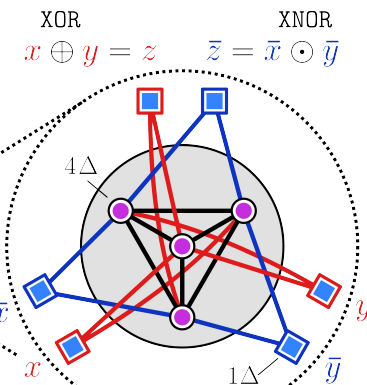
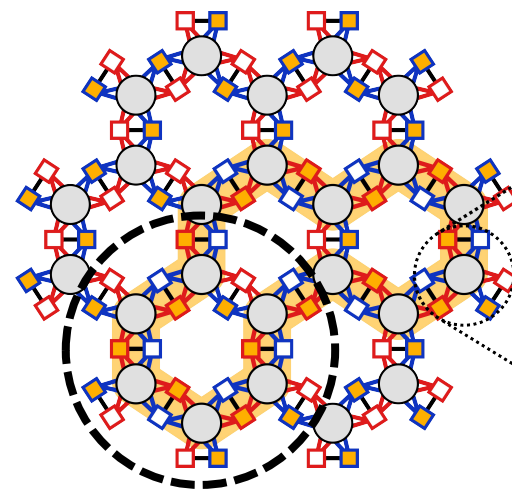


Fully symmetric  
One qubit per edge



Open strings still allowed ...

Enforce XOR/XNOR on sites



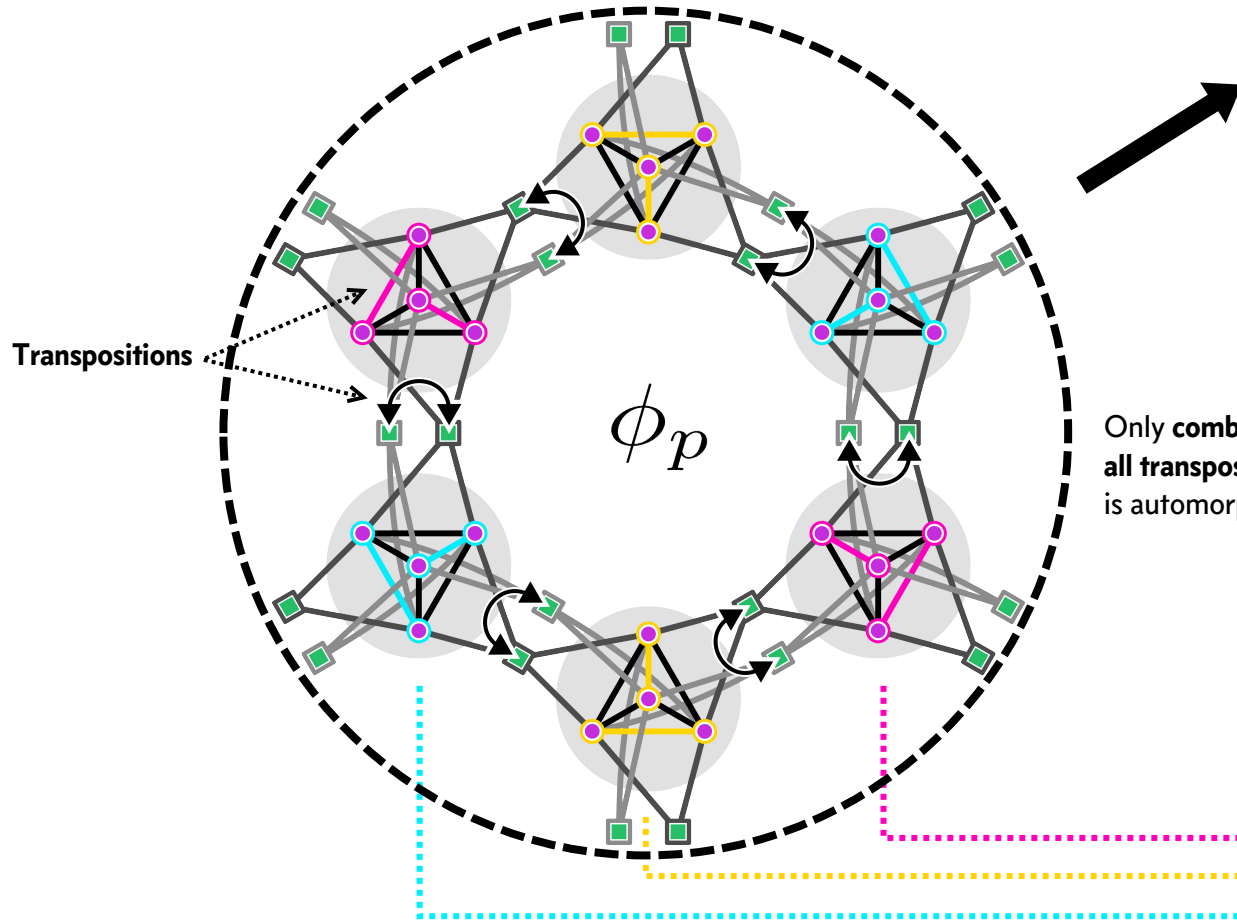
FSU gate has all required properties!



$$\text{String} \leftrightarrow |10\rangle_e \quad \text{and} \quad \text{Empty} \leftrightarrow |01\rangle_e$$

# Local Graph Automorphisms

Graph automorphisms of tessellation?



Transpositions

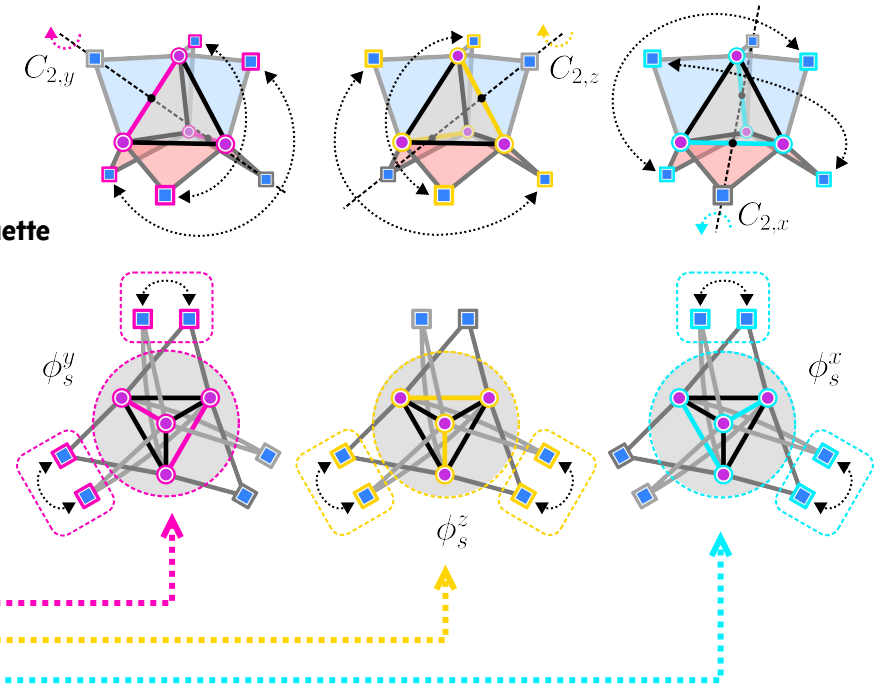
$\phi_p$

Only combination of all transpositions around plaquette is automorphism!

→ **Plaquette symmetries**  $U_p |\mathbf{n}\rangle := |\phi_p \cdot \mathbf{n}\rangle$

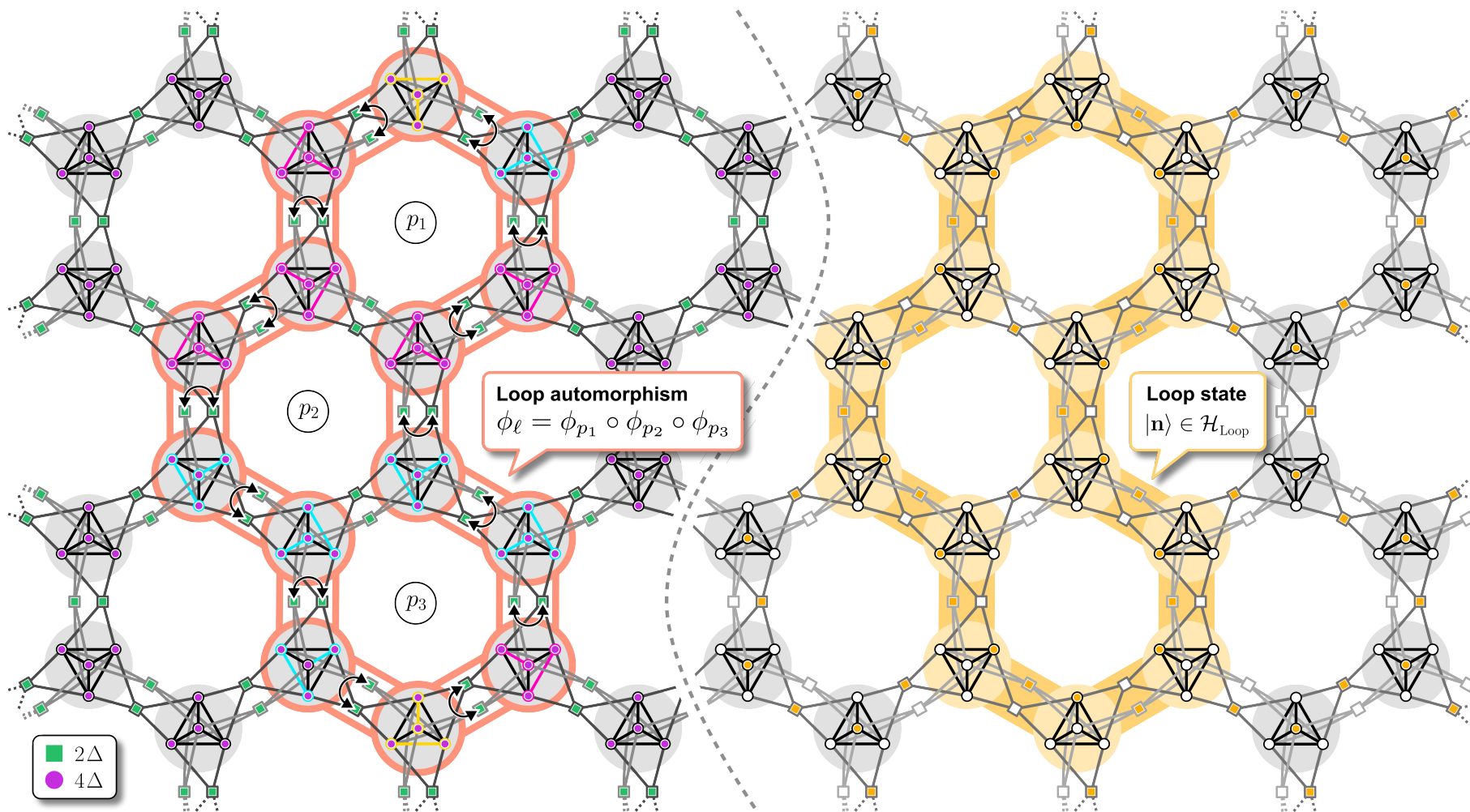
$$[U_p, U_q] = 0 \quad \text{and} \quad [H_{\text{Loop}}, U_p] = 0$$

$$U_p^2 = \mathbb{1} \quad U_p^\dagger = U_p$$



→ **Plaquette automorphisms**

# Putting it all together



→ Loop automorphisms

→ Loop states/patterns

Not a unit disk embedding!

But:  
Extended version can be embedded in quasi-2D. 😊

Ground states  
=  
Loop patterns

No-loop state &  
All loop automorphisms  
→ All loop patterns



Fully symmetric!  
😊

# Topological Order?

Fully symmetric



Unique ground state  
(finite system)

$$|\Omega\rangle = \underbrace{\lambda(\Omega) \sum_{\mathbf{n} \in L_{\text{Loop}}} |\mathbf{n}\rangle}_{=:\Lambda|\Omega_0\rangle} + \sum_{d \geq 1} \underbrace{\left(\frac{\Omega}{\Delta E}\right)^d \sum_{\mathbf{n} \in L_{\text{Loop}}^d} \eta_{\mathbf{n}}(\Omega) |\mathbf{n}\rangle}_{=:\sqrt{1-\Lambda^2}|\Omega_\eta\rangle}$$

No fixpoint  
Hamiltonian possible

Valid for arbitrary  
boundary conditions

Topologically  
ordered



Valid for arbitrary  $\Omega$

Can **destroy**  
topological order!



# Topological Order!

Unphysical!

Auxiliary Hamiltonian

$$\tilde{H}(\Omega, \omega) := H_{\text{Loop}}(\Omega) + \frac{\omega}{2} \sum_p (\mathbb{1} - U_p)$$

Topologically ordered!

$$\Omega = 0 < \omega$$

$$0 < |\Omega| \ll \omega$$

$$\sum_{n \in L_{\text{Loop}}} |n\rangle \propto |\Omega_0\rangle \xrightarrow[\omega > 0]{\Omega = 0 \rightarrow \Omega \neq 0} |\Omega^*\rangle \longrightarrow |\Omega^*\rangle \xrightarrow[\Omega \neq 0]{\omega > 0 \rightarrow \omega = 0} |\Omega\rangle = |\Omega^*\rangle$$

**Gap stability**  
(LTQO & Frustration-free)

**Local symmetry**  
 $[\tilde{H}(\Omega, \omega), U_p] = 0$

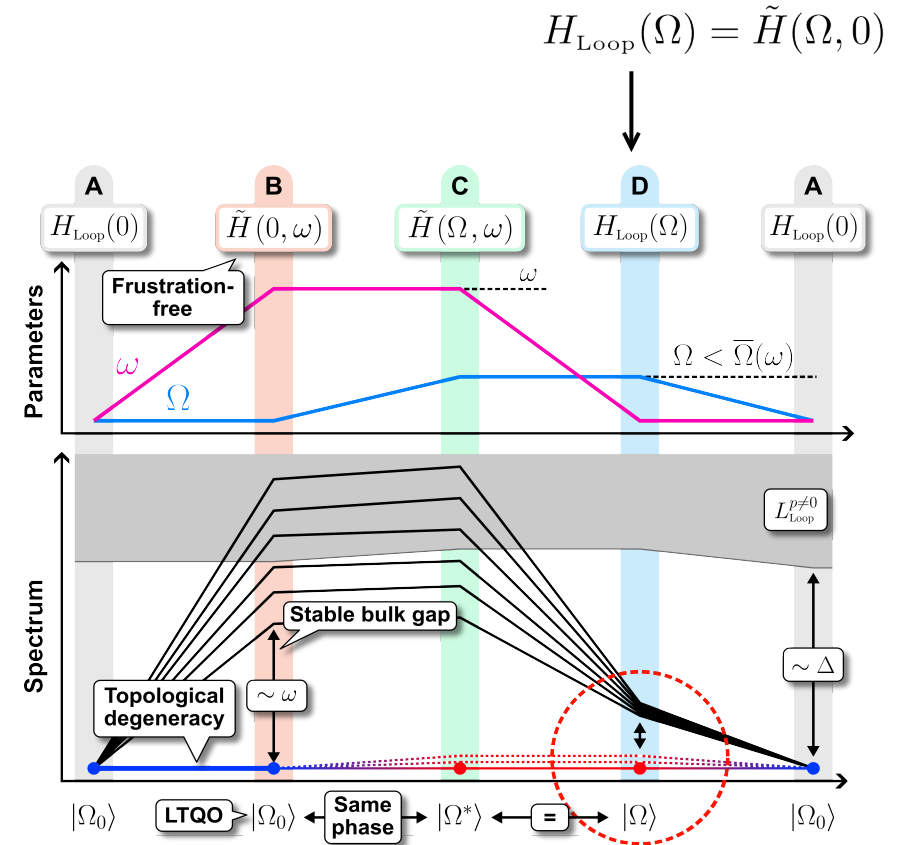
**Summary**

Topologically ordered!

$$|\Omega_0\rangle \sim |\Omega^*\rangle = |\Omega\rangle = \Lambda |\Omega_0\rangle + \sqrt{1 - \Lambda^2} |\Omega_\eta\rangle$$

Same phase (over  $|\Omega_0\rangle \sim |\Omega^*\rangle$ )  
Same state (under  $|\Omega^*\rangle = |\Omega\rangle$ )

Does not destroy topological order!



Open question: **Gap in charge sector?**

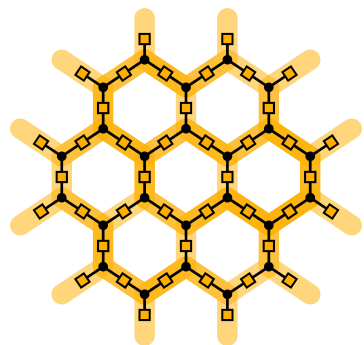
Stability of frustration-free Hamiltonians

S. Michalakis and J. P. Zwolak

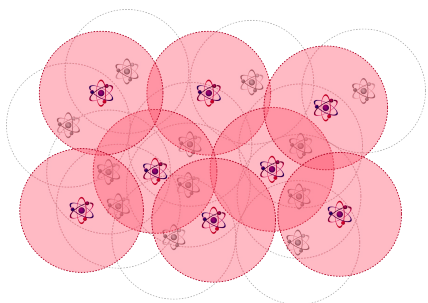
Communications in Mathematical Physics **322**(2), 277 (2013), doi:10.1007/s00220-013-1762-6

# Summary & Outlook

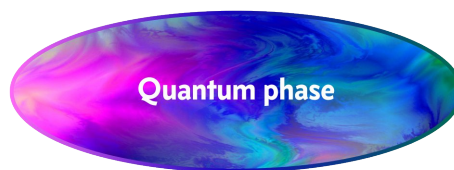
Toric code topological order



Rydberg platform

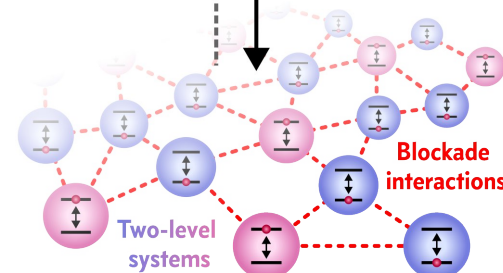


Goal

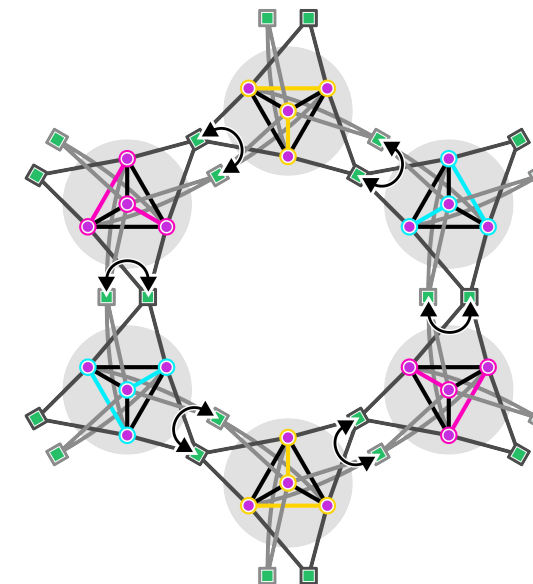


Construction?

Toolbox



Topological Order with  
Fully-Symmetric  
Blockade Structures



Existence of charge gap?

Non-Abelian topological orders?

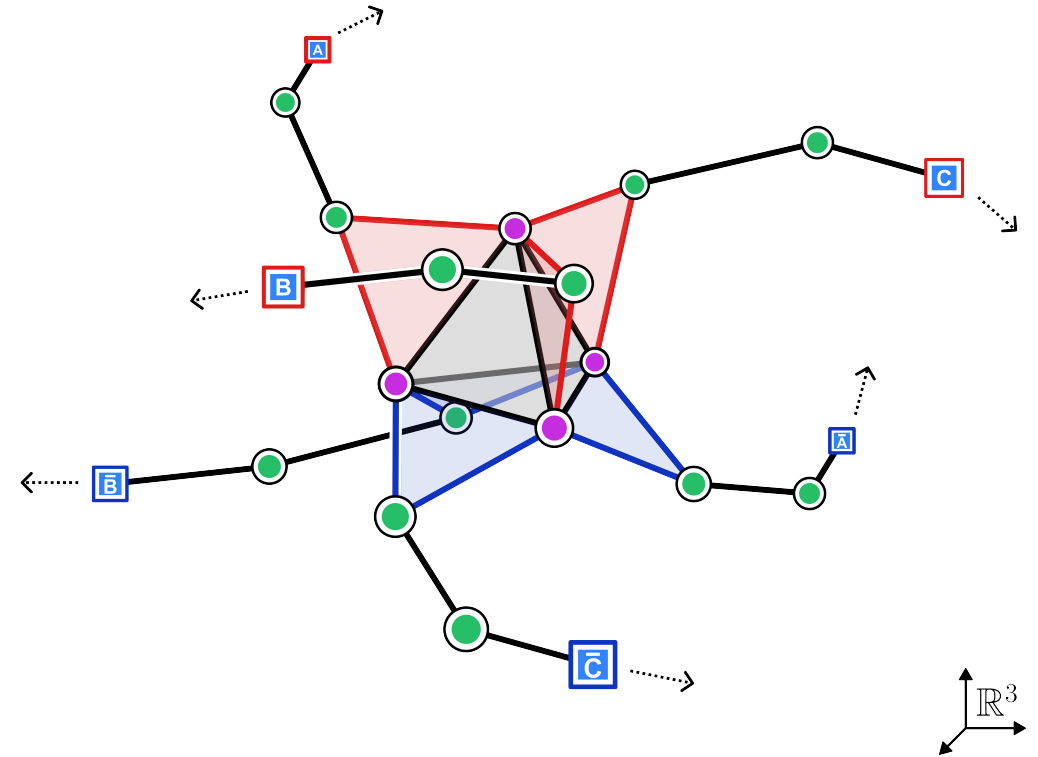
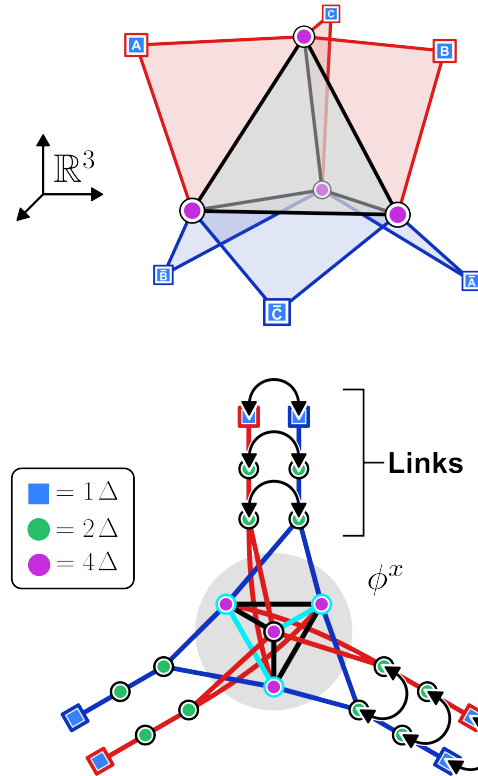
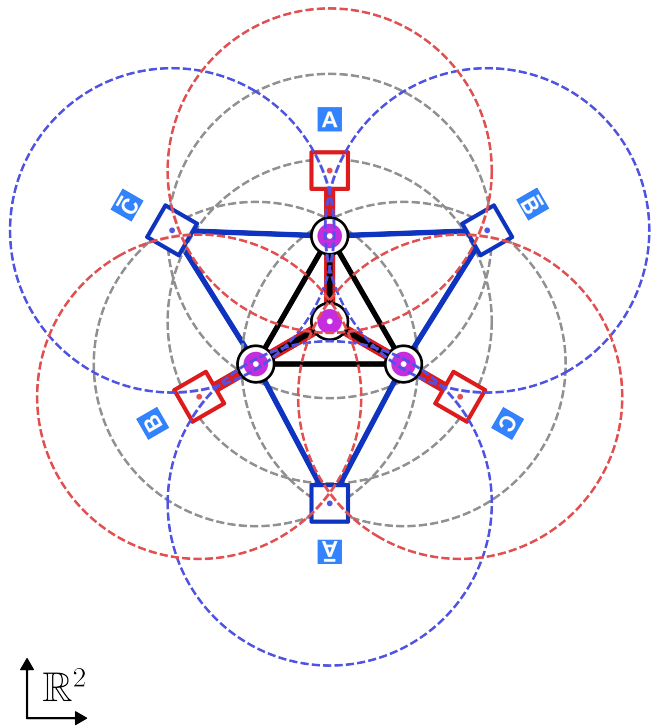
Embeddings?

Size of charge gap?

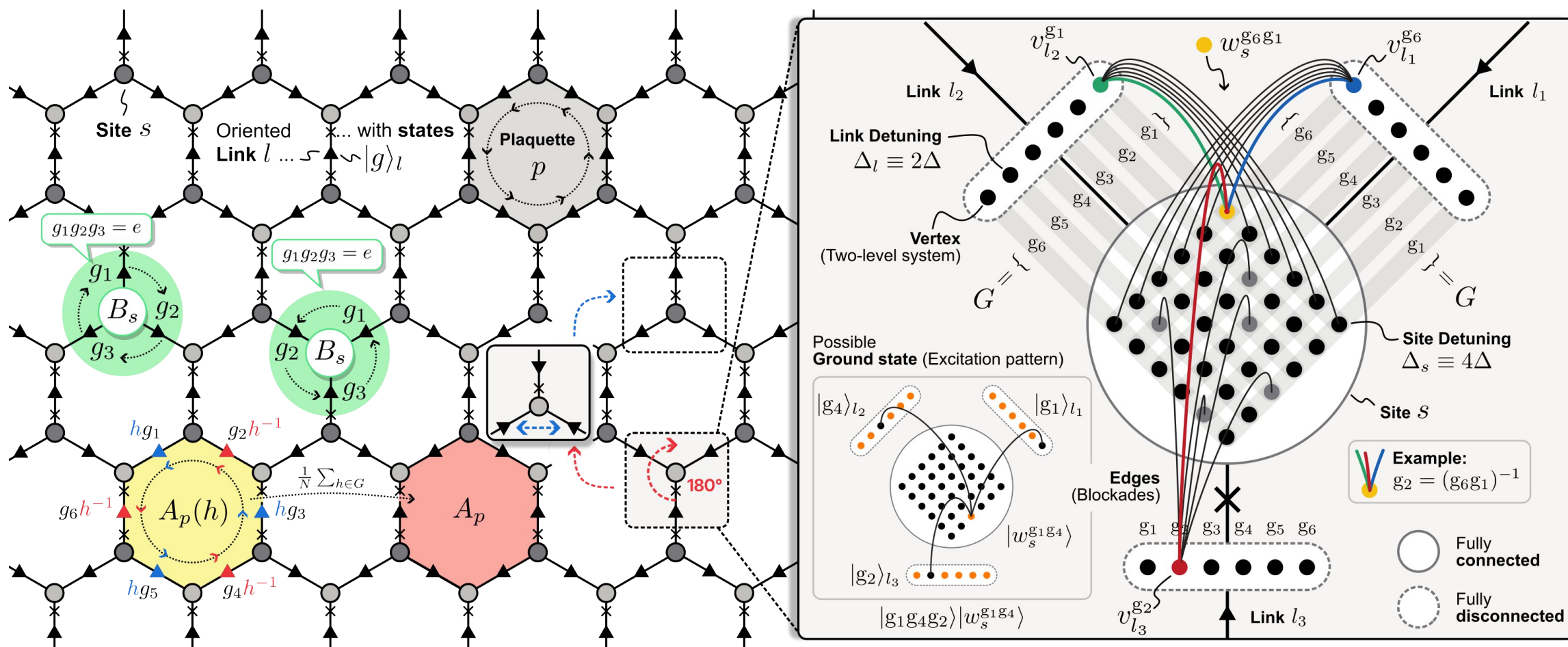
Effects of van der Waals tails?

Non-equal-weight condensates?

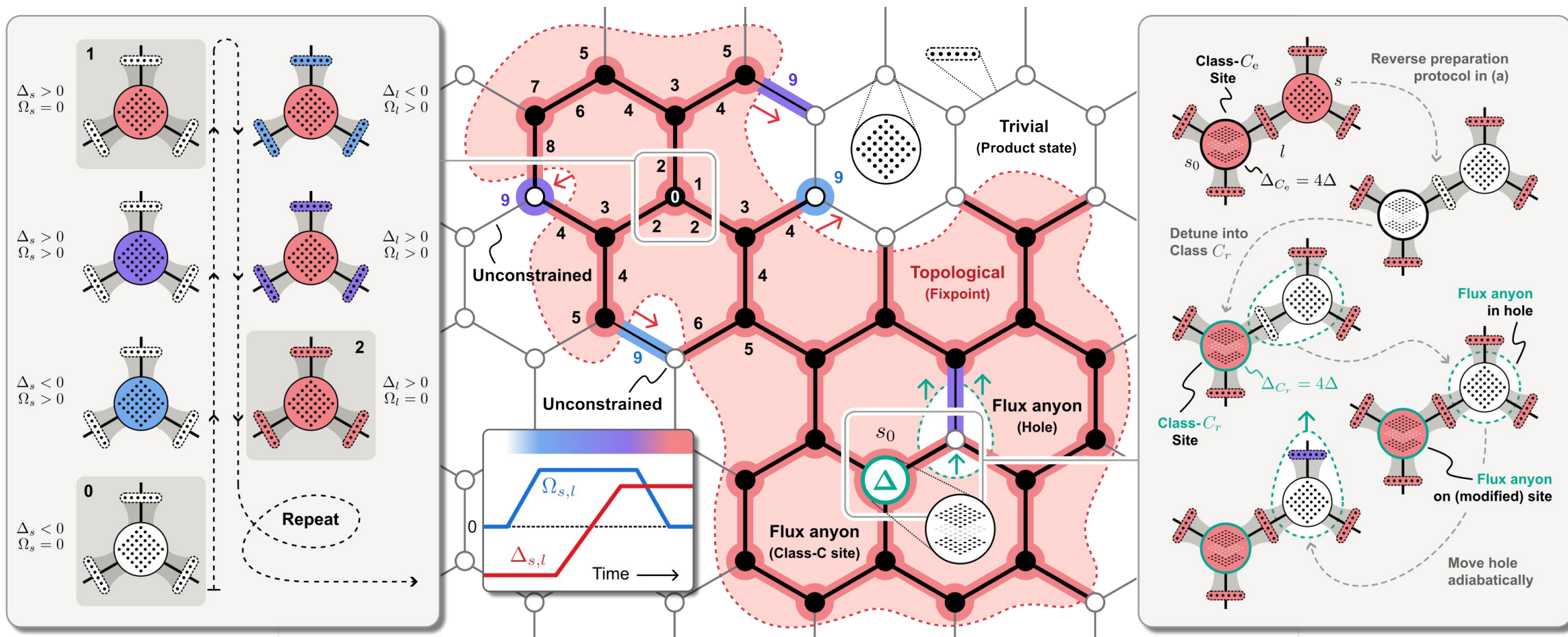
# Embedding of (extended) FSU vertices



# Generalization to Quantum Doubles



# Preparation



# Braiding

