Universität Stuttgart

One-Dimensional Topological States of Synthetic Quantum Matter

Many-Body Ground States & Non-Abelian Statistics

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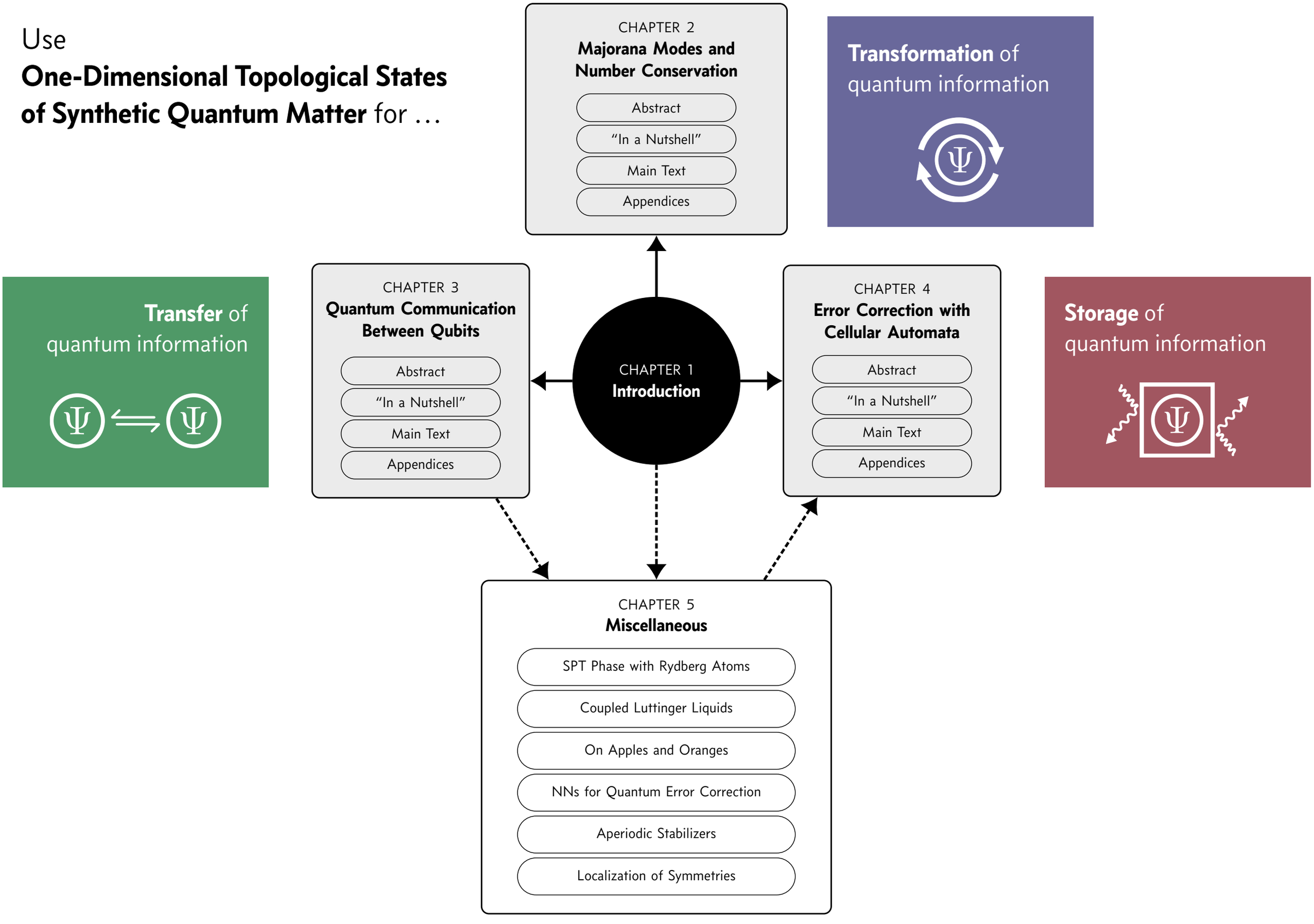
Use
One-Dimensional Topological States of Synthetic Quantum Matter for …



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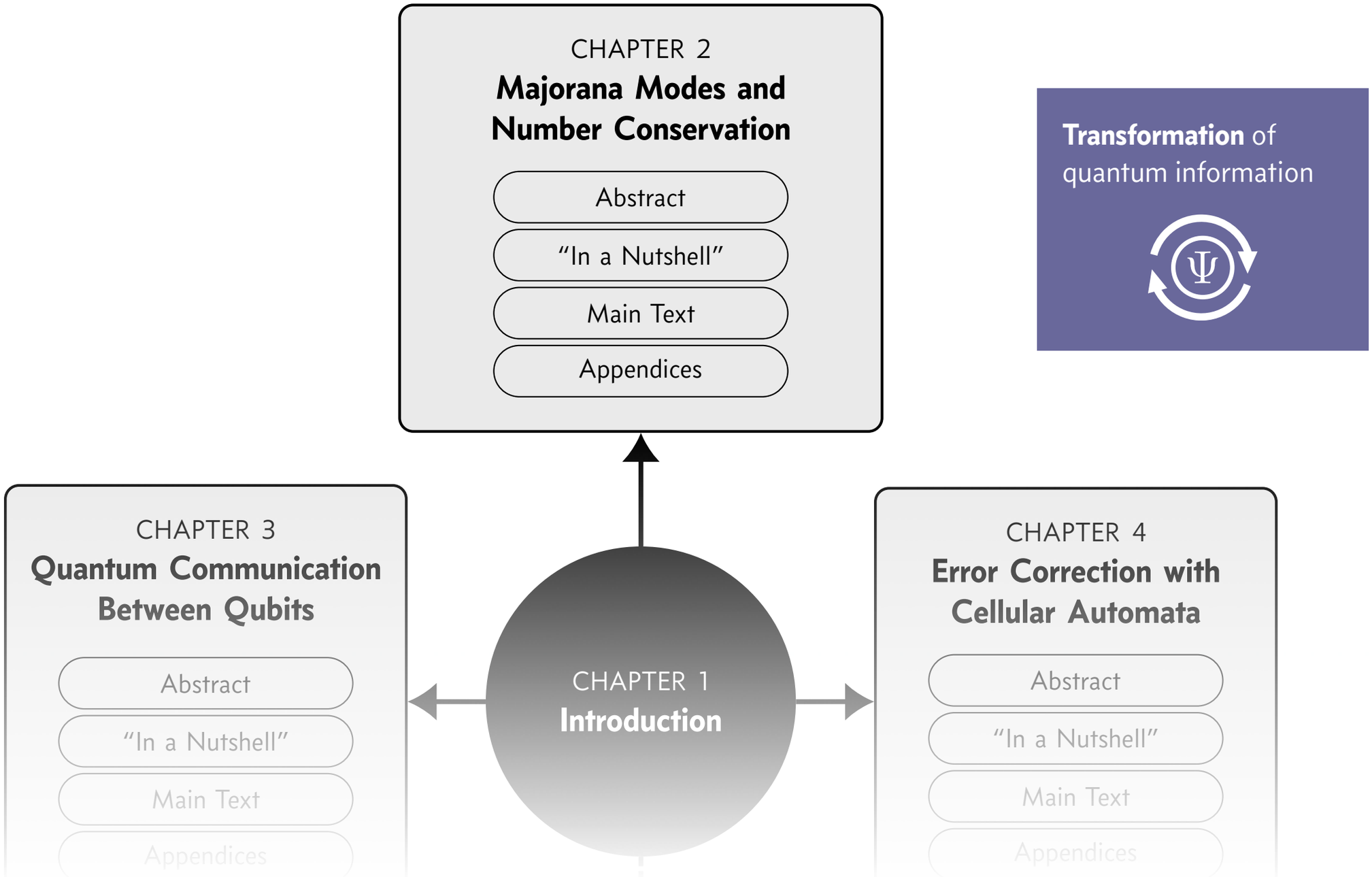

Transformation of quantum information

CHAPTER 2

Majorana Modes in a Number-Conserving Theory

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PART 1

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Motivation & Goal

Quantum Information Theory ⋯⋯⋯⋯⋯⋯⋯⋯⋯⋯➤ **Condensed Matter Theory**

Qubit

$$|\Psi\rangle = \alpha\,|\oplus\rangle + \beta\,|\ominus\rangle$$

but:

Perturbations

$$H_{\text{pert}} = \begin{pmatrix} 0 & c \\ c^* & a \end{pmatrix}$$

Depolarization Dephasing

*We are looking for a **1D** system …*

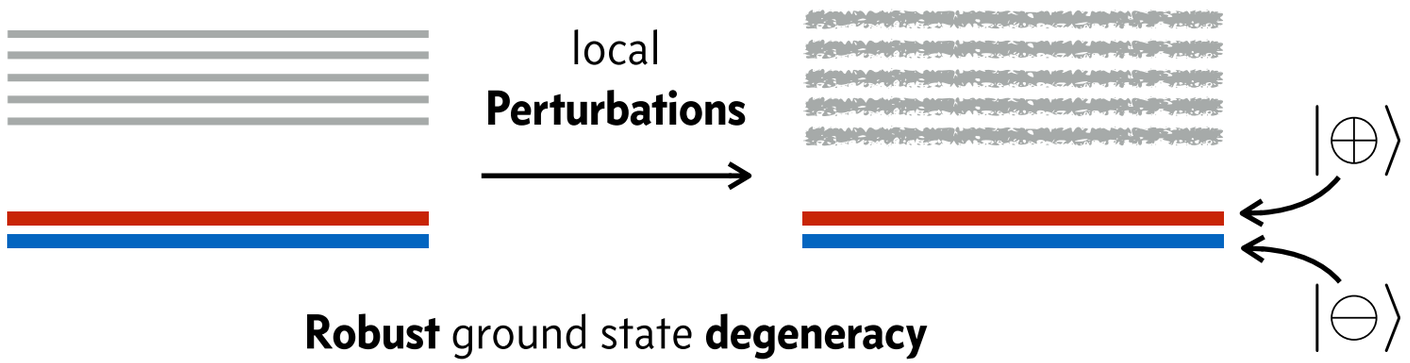
Local Hamiltonian

$$H = \sum_i \ldots ?$$

$$\left|\ \uparrow\uparrow\downarrow\uparrow\downarrow\downarrow\downarrow\uparrow\downarrow\uparrow\uparrow\downarrow 0010110101\ \right\rangle \in \mathcal{H}$$

1D system of bosons/fermions (?)

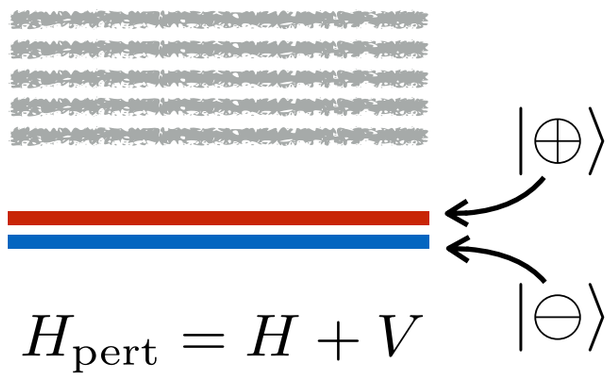
… with the feature:



Robust ground state **degeneracy**

Topological Order

local **Perturbations** V



Condition for **robust** ground state **degeneracy**:

$$H_{\text{eff}} = \begin{pmatrix} \langle\oplus|V|\oplus\rangle & \langle\oplus|V|\ominus\rangle \\ \langle\ominus|V|\oplus\rangle & \langle\ominus|V|\ominus\rangle \end{pmatrix} \overset{!}{\sim} \mathbb{1}$$

Definition of **Topological Order**

A finite set of states $\mathcal{G} = \{|g_i\rangle\}$ is **topologically ordered** if

$$\langle g_i|V|g_j\rangle \simeq v\,\delta_{ij}$$

for **all local perturbations** V

Z. Nussinov and G. Ortiz, *A symmetry principle for topological quantum order*, Annals of Physics **324,** 977–1057 (2009)

Spins?

$$H_{\text{eff}} = \begin{pmatrix} \langle\oplus|V|\oplus\rangle & \langle\oplus|V|\ominus\rangle \\ \langle\ominus|V|\oplus\rangle & \langle\ominus|V|\ominus\rangle \end{pmatrix} \overset{!}{\sim} \mathbb{1}$$

	$\langle\oplus\|V\|\ominus\rangle = 0\,?$	$\langle\oplus\|V\|\oplus\rangle = \langle\ominus\|V\|\ominus\rangle\,?$
First try ($\sigma^z\sigma^z$-Ising model GS) : $\mathcal{G} = \begin{cases} \|\oplus\rangle = \|\uparrow\uparrow\uparrow\ \dots\ \uparrow\rangle \\ \|\ominus\rangle = \|\downarrow\downarrow\downarrow\ \dots\ \downarrow\rangle \end{cases}$	✓	✗ apply σ^z on **arbitrary** site
Second try ($\sigma^x\sigma^x$-Ising model GS) : $\mathcal{G} = \begin{cases} \|\oplus\rangle = \sum_{\mathbf{s}\in\mathcal{A}_+} \|\mathbf{s}\rangle \\ \|\ominus\rangle = \sum_{\mathbf{s}\in\mathcal{A}_-} \|\mathbf{s}\rangle \end{cases}$	✗ apply σ^x on **arbitrary** site	✓

$\mathcal{A}_\pm = \{$ configurations with **even/odd** number of down-spins $\}$

Fermions?

$$H_{\text{eff}} = \begin{pmatrix} \langle\oplus|V|\oplus\rangle & \langle\oplus|V|\ominus\rangle \\ \langle\ominus|V|\oplus\rangle & \langle\ominus|V|\ominus\rangle \end{pmatrix} \overset{!}{\sim} \mathbb{1}$$

	$\langle\oplus\|V\|\ominus\rangle = 0\,?$	$\langle\oplus\|V\|\oplus\rangle = \langle\ominus\|V\|\ominus\rangle\,?$
First try (Phase separation) : $\mathcal{G} = \{ \|\oplus\rangle = \|0\,0\,0\,\ldots\,0\rangle,\ \|\ominus\rangle = \|1\,1\,1\,\ldots\,1\rangle \}$	✓	✗ apply $c^\dagger c$ on **arbitrary** site
Second try (GS of what?) : $\mathcal{G} = \{ \|\oplus\rangle = \sum_{\mathbf{n}\in\mathcal{A}_+} \|\mathbf{n}\rangle,\ \|\ominus\rangle = \sum_{\mathbf{n}\in\mathcal{A}_-} \|\mathbf{n}\rangle \}$	✓ (✗)	✓

$\mathcal{A}_\pm = \{$ configurations with **even/odd** number of fermions $\}$ **Fermion parity!**

Fermions!

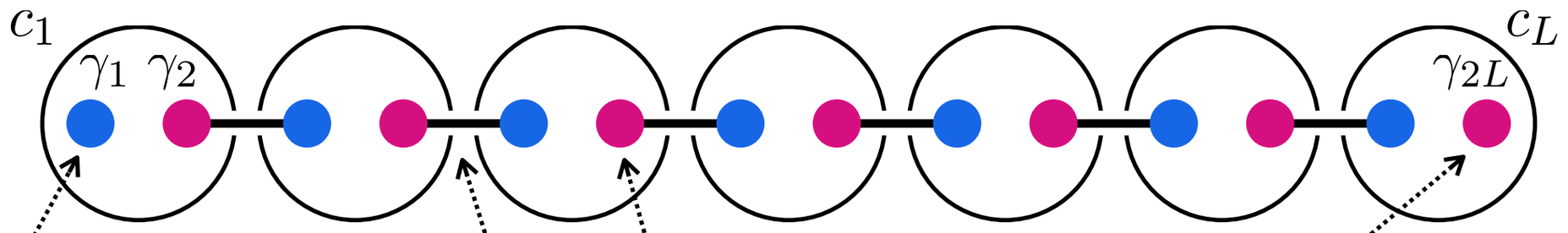
„**Even-parity** equal-weight superposition“ „**Odd-parity** equal-weight superposition“

$$|\oplus\rangle = \sum_{\mathbf{n}\in\mathcal{A}_+} |\mathbf{n}\rangle \quad \longleftarrow\ ?\ \longrightarrow \quad |\ominus\rangle = \sum_{\mathbf{n}\in\mathcal{A}_-} |\mathbf{n}\rangle$$

Spins: $\sigma_i^x \rightarrow$, $\sigma_i^y \rightarrow$ Fermions:

$$c_i^\dagger + c_i \equiv \gamma_{2i-1} \quad \textbf{Majorana} \text{ mode}$$
$$i(c_i^\dagger - c_i) \equiv \gamma_{2i} \quad \textbf{Majorana} \text{ mode}$$

$$\left.\begin{aligned} \{\gamma_i, \gamma_j\} &= 2\delta_{ij} \\ \gamma_i^\dagger &= \gamma_i \end{aligned}\right\}$$



$$\gamma_1|\oplus\rangle = |\ominus\rangle \qquad \langle\ominus|\gamma_j|\oplus\rangle = 0 \qquad \gamma_{2L}|\oplus\rangle = +i|\ominus\rangle$$
$$\gamma_1|\ominus\rangle = |\oplus\rangle \qquad i\gamma_{2j}\gamma_{2j+1}|\bigcirc\rangle = -|\bigcirc\rangle \qquad \gamma_{2L}|\ominus\rangle = -i|\oplus\rangle$$

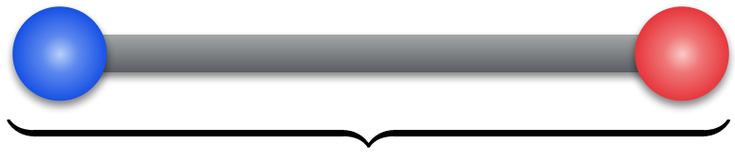
with the fermion ordering $|\mathbf{n}\rangle = (c_1^\dagger)^{n_1}(c_2^\dagger)^{n_2}\ldots(c_L^\dagger)^{n_L}|0\,0\,\ldots\,0\rangle$

The Majorana Chain

$$\gamma_1|\oplus\rangle = |\ominus\rangle \qquad \langle\ominus|\gamma_j|\oplus\rangle = 0 \qquad \gamma_{2L}|\oplus\rangle = +i|\ominus\rangle$$
$$\gamma_1|\ominus\rangle = |\oplus\rangle \qquad i\gamma_{2j}\gamma_{2j+1}|\bigcirc\rangle = -|\bigcirc\rangle \qquad \gamma_{2L}|\ominus\rangle = -i|\oplus\rangle$$

Ground states form a **qubit** split in „**half**“ :

$$\gamma_1 \triangleq \sigma^x \qquad \gamma_{2L} \triangleq \sigma^y$$



$$-i\gamma_1\gamma_{2L} \triangleq \sigma^z$$

Collides with locality / causality!

All observables are **parity-symmetric** :

Parity superselection

Gapped Hamiltonian with ground states $|\oplus\rangle, |\ominus\rangle$:

$$H_{\mathrm{MC}} = w\sum_{j=1}^{L-1}\left(i\,\gamma_{2j}\gamma_{2j+1}\right)$$
$$= -w\sum_j\left(c_j^\dagger c_{j+1} - c_j c_{j+1} + \mathrm{h.c.}\right)$$

1D p-wave superconductor
(mean field)

Topological phase of the

Majorana chain

(flat band regime)

A. Kitaev, *Unpaired Majorana fermions in quantum wires*, Phys.-Usp. **44**, 131 (2001)

Why this Model?

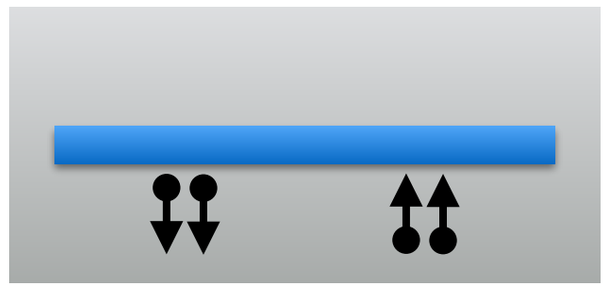
The Majorana Chain

Majorana chain

$$H_{\mathrm{MC}} = \sum_i \ldots + c_i^\dagger c_{i+1}^\dagger + c_{i+1} c_i + \ldots$$

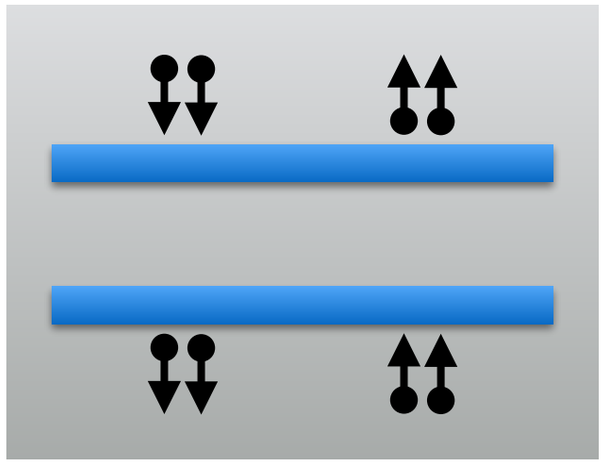
BUT … mean field ?!

3D Bulk Superconductor



Quantum wire

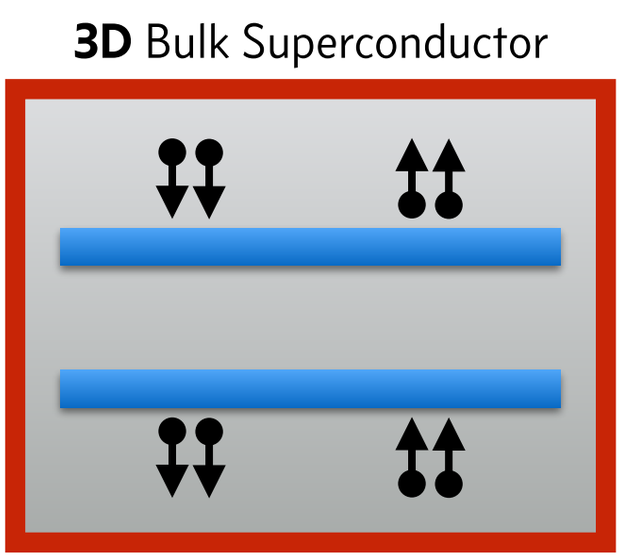
3D Bulk Superconductor



2 x Quantum wires

BUT … parity superselection ?!

The Majorana Chain



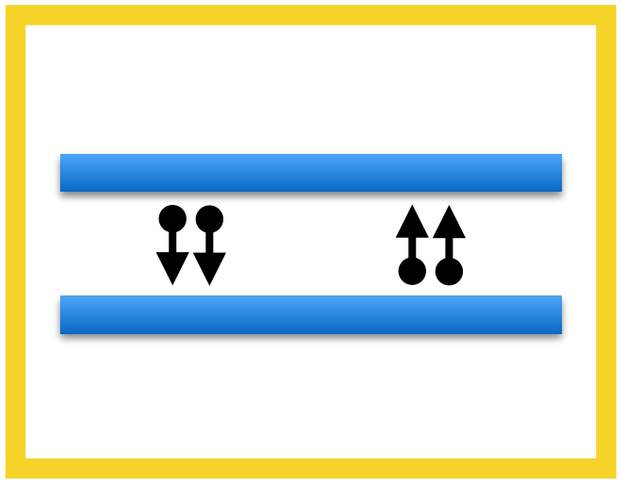
2 x Quantum wires

Fix **global** parity

& encode qubit in **subchain parity**

The Double Chain

Can we get rid of the bulk superconductor?



Double wire/chain

Fix **global particle number**

& encode qubit in **subchain parity**

The Double Chain Ground States ?

Fermion ordering:

$$|\mathbf{m} = (\mathbf{u}, \mathbf{l})\rangle = (a_1^\dagger)^{u_1} \dots (a_L^\dagger)^{u_L} \, (b_1^\dagger)^{l_1} \dots (b_L^\dagger)^{l_L} |0 \dots 0\rangle$$

For each **particle number** N :

$$|\ominus\rangle = \sum_{\mathbf{m} \in \mathcal{A}_-^N} |\mathbf{m}\rangle$$

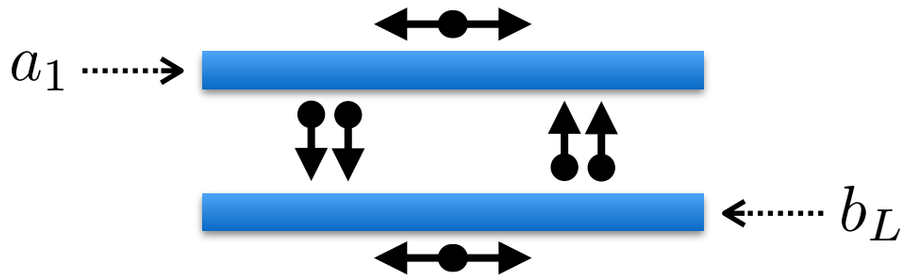
$$|\oplus\rangle = \sum_{\mathbf{m} \in \mathcal{A}_+^N} |\mathbf{m}\rangle$$

$$\mathcal{A}_\pm^N = \left\{ \begin{array}{c} \text{configurations of } N \text{ fermions with} \\ \textbf{even/odd} \text{ number on } \textbf{upper subchain} \end{array} \right\}$$

NL and HPB, *Topological states in a microscopic model of interacting fermions*, Phys. Rev. B **92**, 041118(R) (2015)

The Double Chain

Hamiltonian ?



intra-chain **single-particle hopping** & attraction

$$H_{\mathrm{DC}} = \sum_{x\in\{a,b\}} \sum_i \left[-x_i^\dagger x_{i+1} + n_i^x (1 - n_{i+1}^x) + \{i \leftrightarrow i+1\} \right]$$

$$+ \sum_i \left[a_i^\dagger a_{i+1}^\dagger b_i b_{i+1} + n_i^a n_{i+1}^a (1 - n_i^b)(1 - n_{i+1}^b) + \{a \leftrightarrow b\} \right]$$

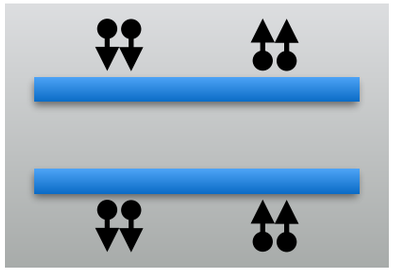
inter-chain **pair hopping** & attraction

Interactions make Hamiltonian **positive semi-definite** :

⇒ Zero-energy state = Ground state ⇒ Explicit construction: $|\oplus\rangle, |\ominus\rangle$ ✓

NL and HPB, *Topological states in a microscopic model of interacting fermions*, Phys. Rev. B **92**, 041118(R) (2015)

Comparison

Double chain	Majorana chain(s)
	
Global **particle number** conservation	Global parity symmetry
Interacting Hamiltonian	Non-interacting Hamiltonian
Gapless spectrum / critical point	Gapped spectrum
„**Majorana-like**“ edge states ($\gamma \rightarrow \chi$)	Majorana edge modes (γ)
⋮	⋮
Non-abelian braiding ?	**Non-abelian braiding !**

PART 2

What is it good for?

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Topological Quantum Computation

Anyons Quasiparticles in **2D** systems
(≠ fermions/bosons)

Braiding **Winding** two anyons around each other
(without bringing them close together)

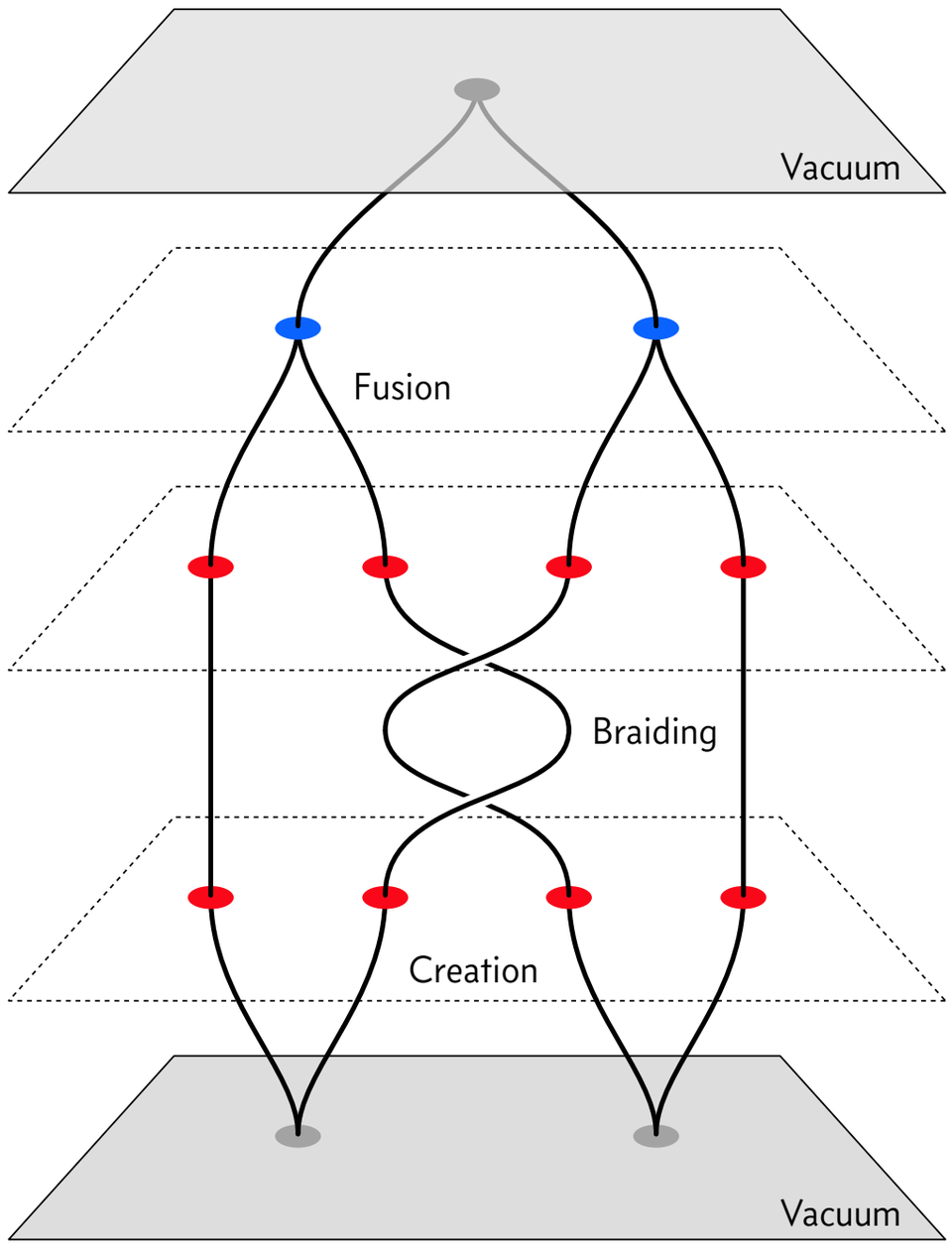
Fusion **Pairing** two anyons and measuring the result
(→ another anyon)

Non-abelian anyons Anyons with **multiple possible** fusion products

Topological Quantum Computation

Initialization → **create** anyons
Gates → **braid** anyons
Measurements → **fuse** anyons

→ Decoherence-free quantum computation



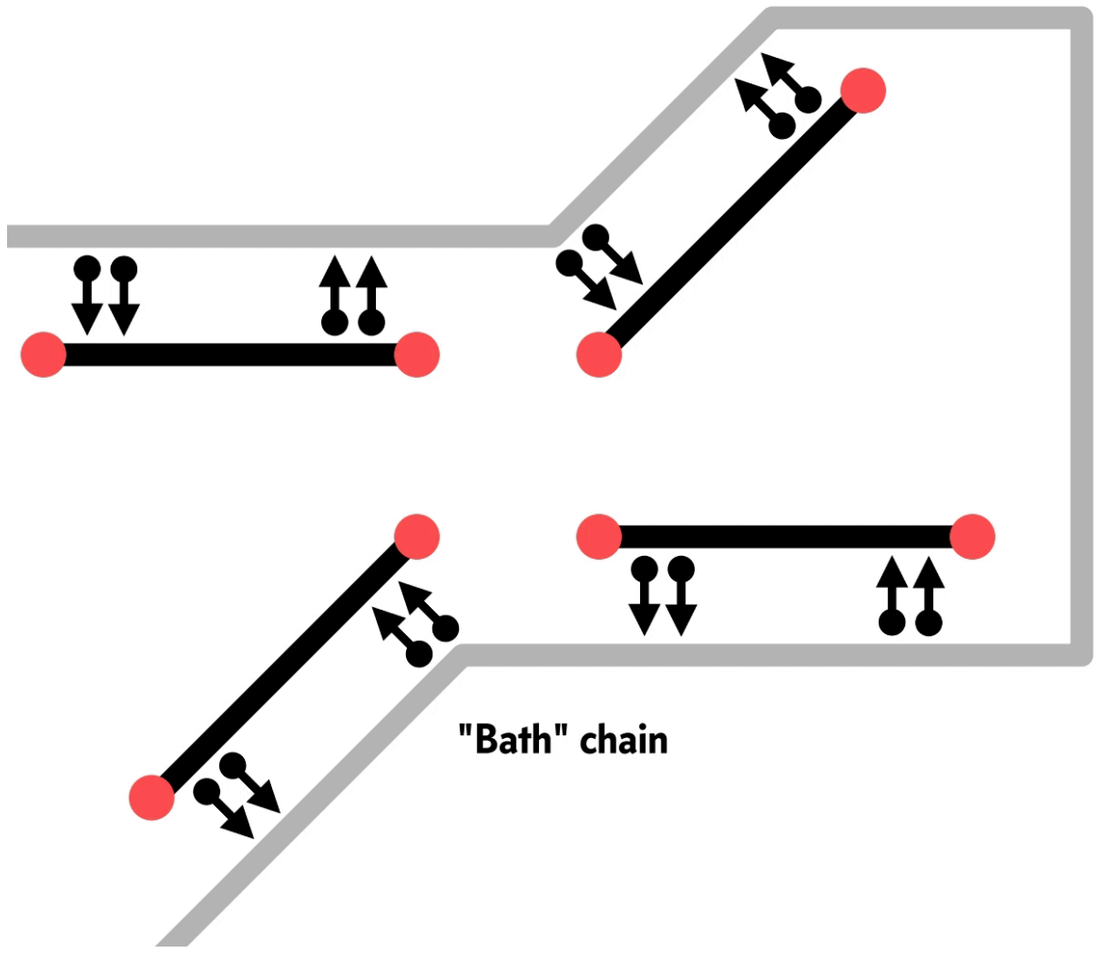
C. Nayak, *et al.*, *Non-Abelian anyons and topological quantum computation*, Rev. Mod. Phys. **80**, 1083 (2008)
A. Kitaev, *Fault-tolerant quantum computation by anyons*, Annals of Physics **303**(1), 2-30 (2003)

Wire Networks

Anyons

Endpoints in **wire networks**

~~Quasiparticles in **2D** systems~~
(≠ fermions/bosons)



Braiding

Winding two anyons around each other
(without bringing them close together)

… by adiabatically **tuning couplings**

Fusion

Pairing two anyons and measuring the result
(→ another anyon)

J. Alicea, *et al.*, *Non-Abelian statistics and topological quantum information processing in 1D wire networks*, Nature Physics **7**, 412–417 (2011)

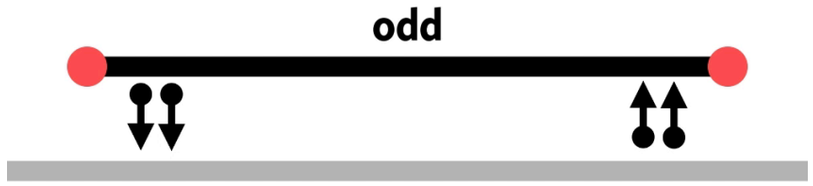
NL and HPB, *Topological states in a microscopic model of interacting fermions*, Phys. Rev. B **92**, 041118(R) (2015)

Wire Networks

Anyons

Endpoints in **wire networks**

~~Quasiparticles in **2D** systems~~
(≠ fermions/bosons)



Braiding

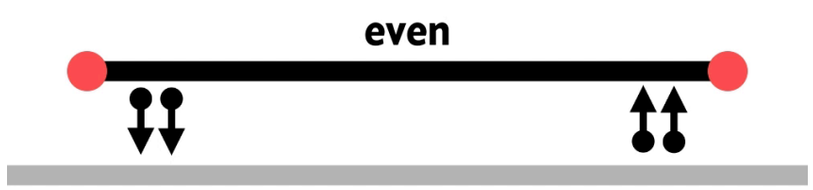
Winding two anyons around each other
(without bringing them close together)

… by adiabatically **tuning couplings**

Fusion

Pairing two anyons and measuring the result
(→ another anyon)

… by **connecting endpoints** of wire segments



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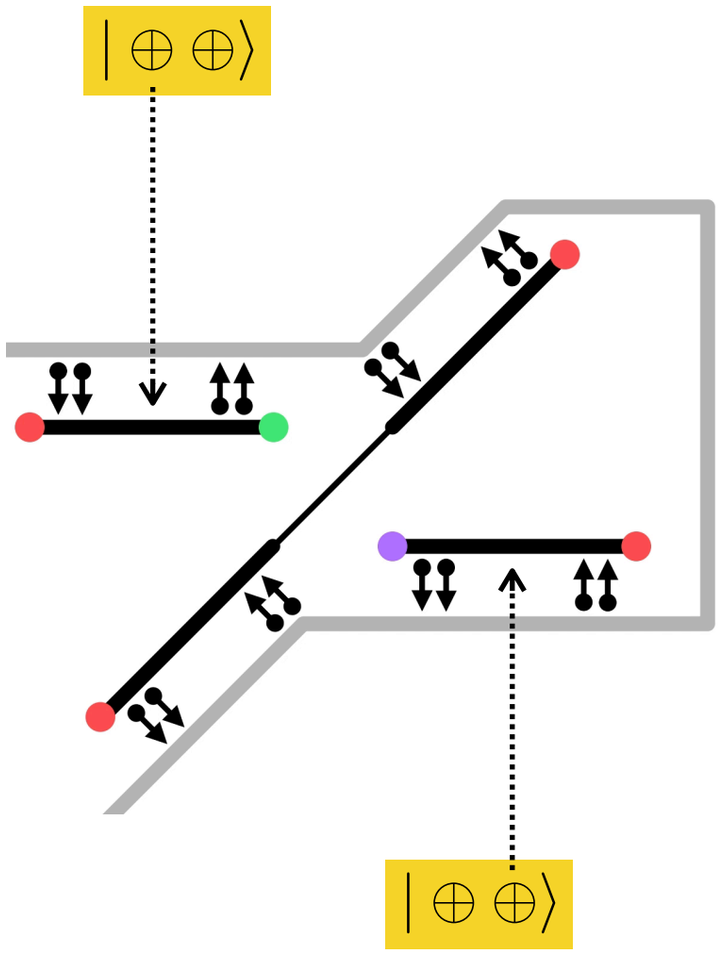
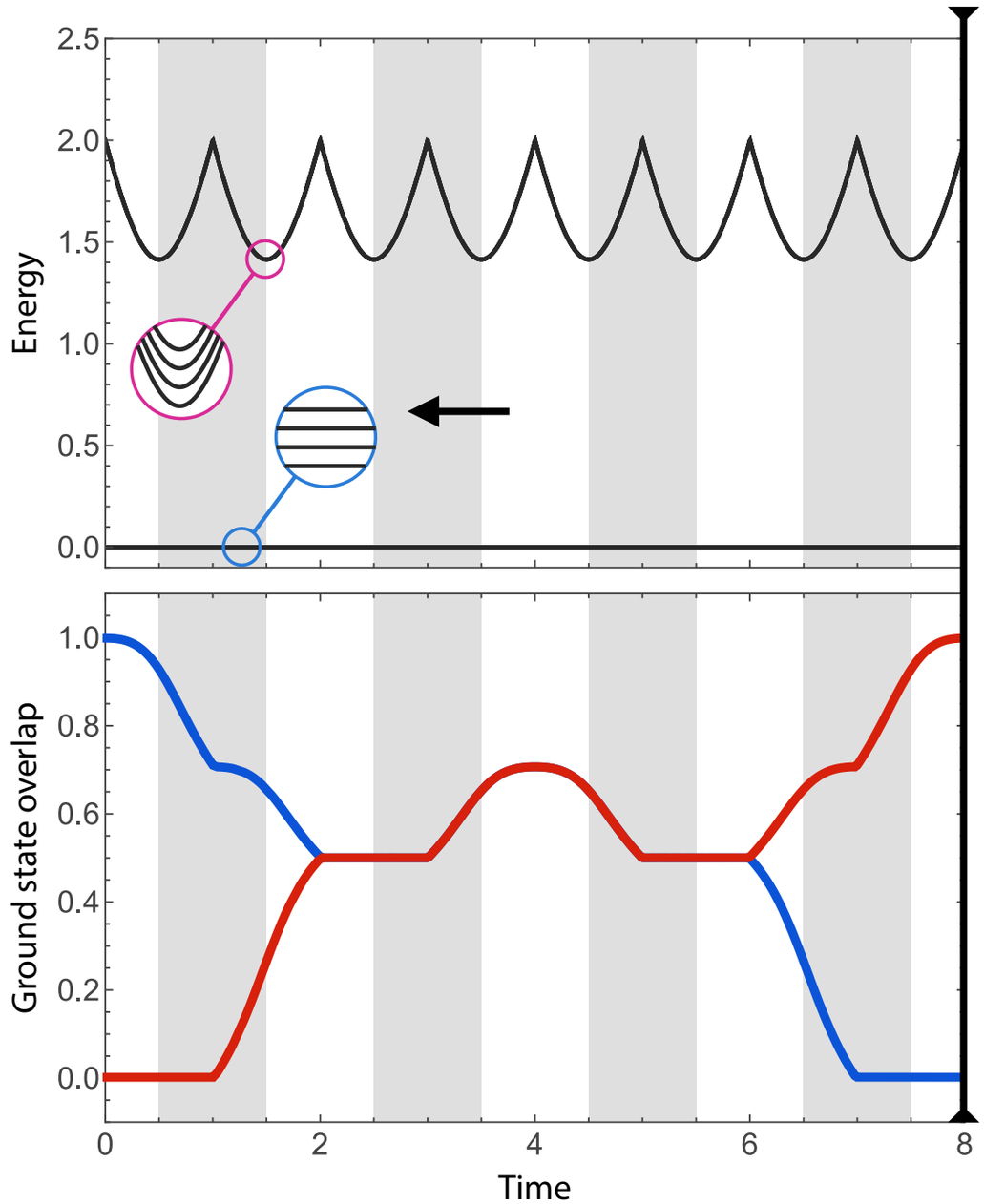
Braiding: Results





8 Ground states

+

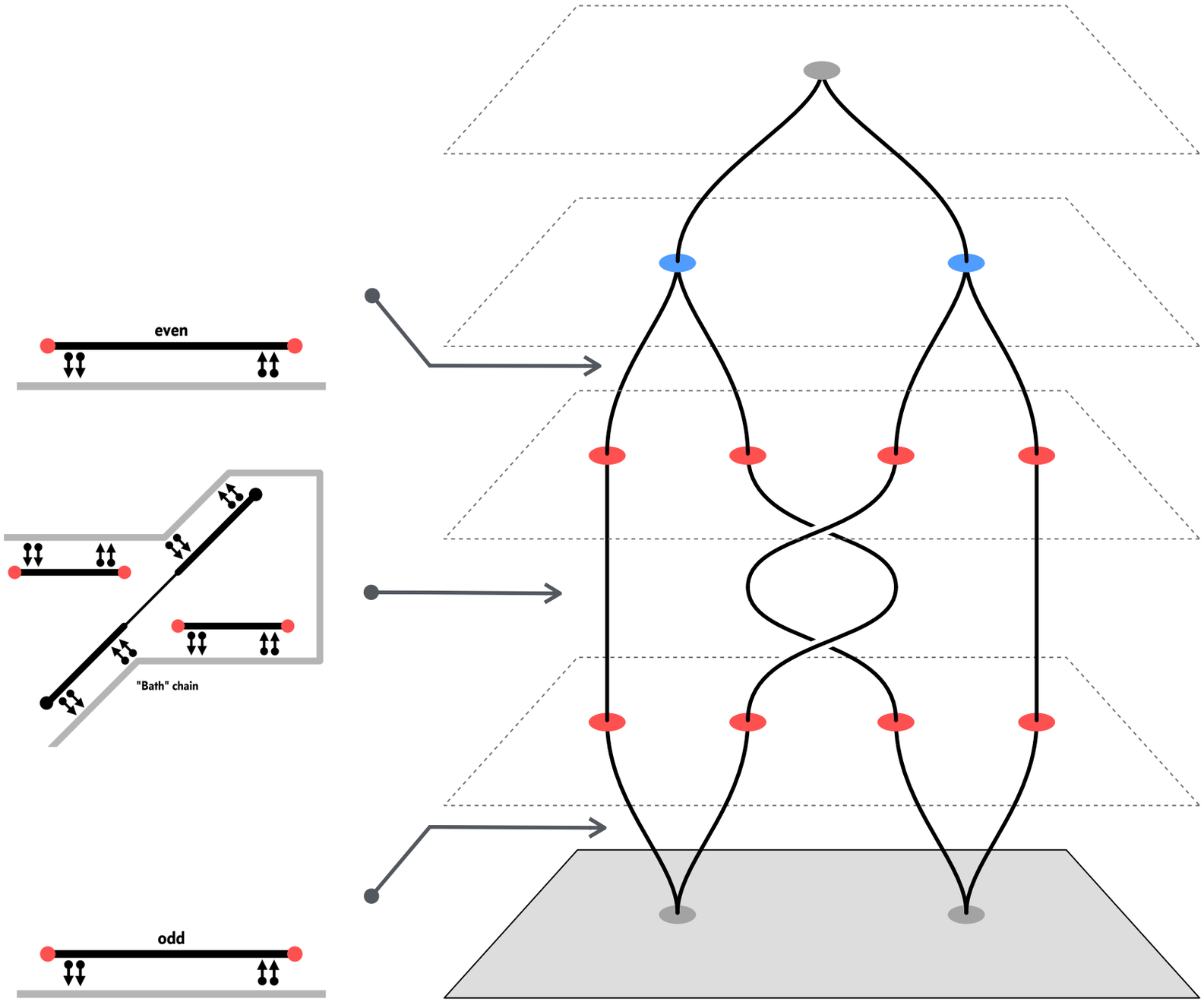
Weak coupling

↓

4 Ground states

NL and HPB, *Topological states in a microscopic model of interacting fermions*, Phys. Rev. B **92**, 041118(R) (2015)

Braiding: Interpretation



Topological **qubit** & **gate**

$$|\oplus\,\oplus\rangle \equiv |1\rangle$$

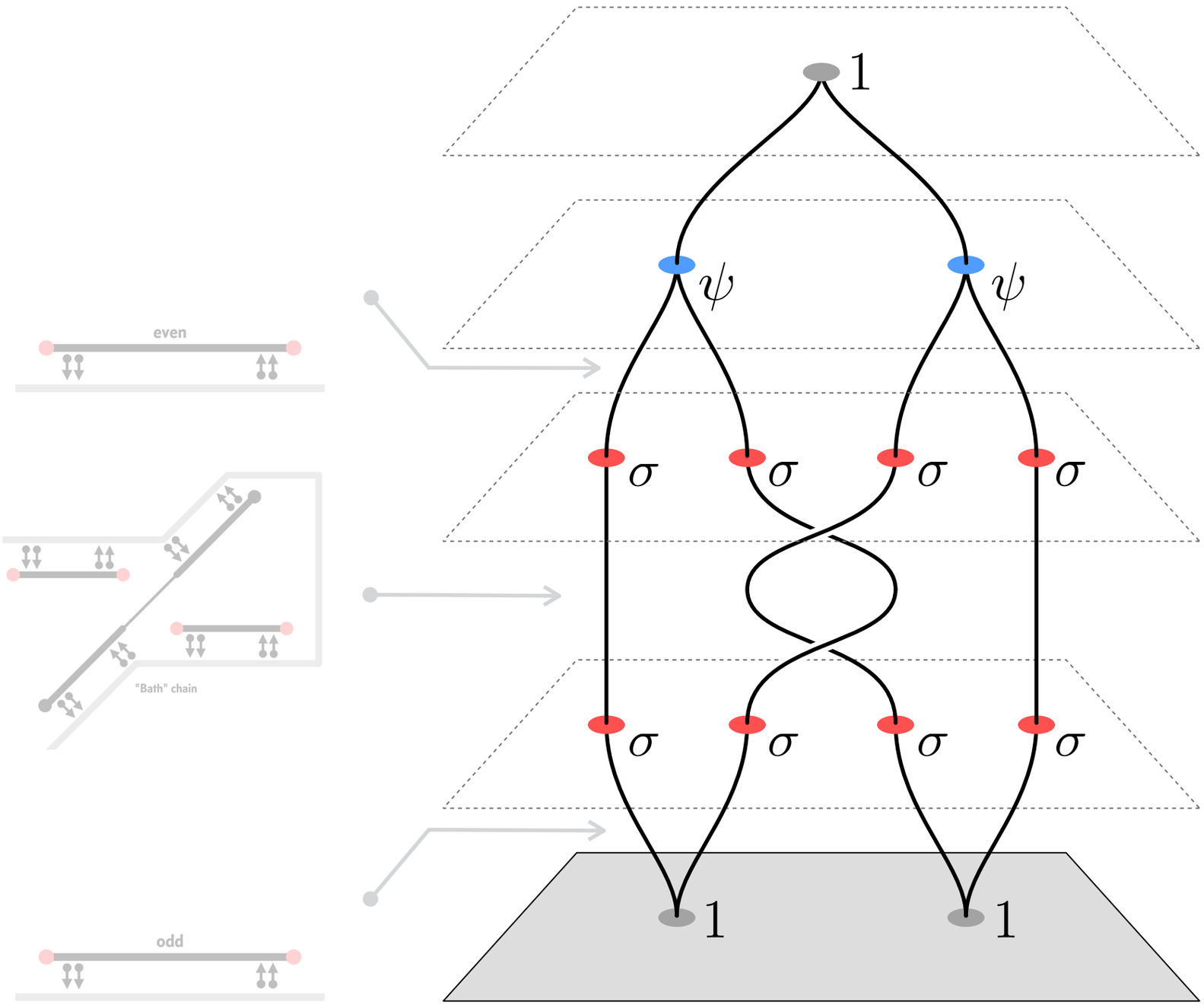
$$\sigma^x$$

$$|\ominus\,\ominus\rangle \equiv |0\rangle$$

A. Kitaev, *Anyons in an exactly solved model and beyond*, Annals of Physics **321**, 2-111 (2006)

NL and HPB, *Topological states in a microscopic model of interacting fermions*, Phys. Rev. B **92**, 041118(R) (2015)

Braiding: Interpretation



Ising anyons

Fusion rules:

$$\sigma \otimes \sigma = 1 \oplus \psi$$
$$\psi \otimes \psi = 1$$
$$\sigma \otimes \psi = \sigma$$
$$\vdots$$

(= Majorana chain)

A. Kitaev, *Anyons in an exactly solved model and beyond*, Annals of Physics **321**, 2-111 (2006)

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Take-Home Message

Why this Model?

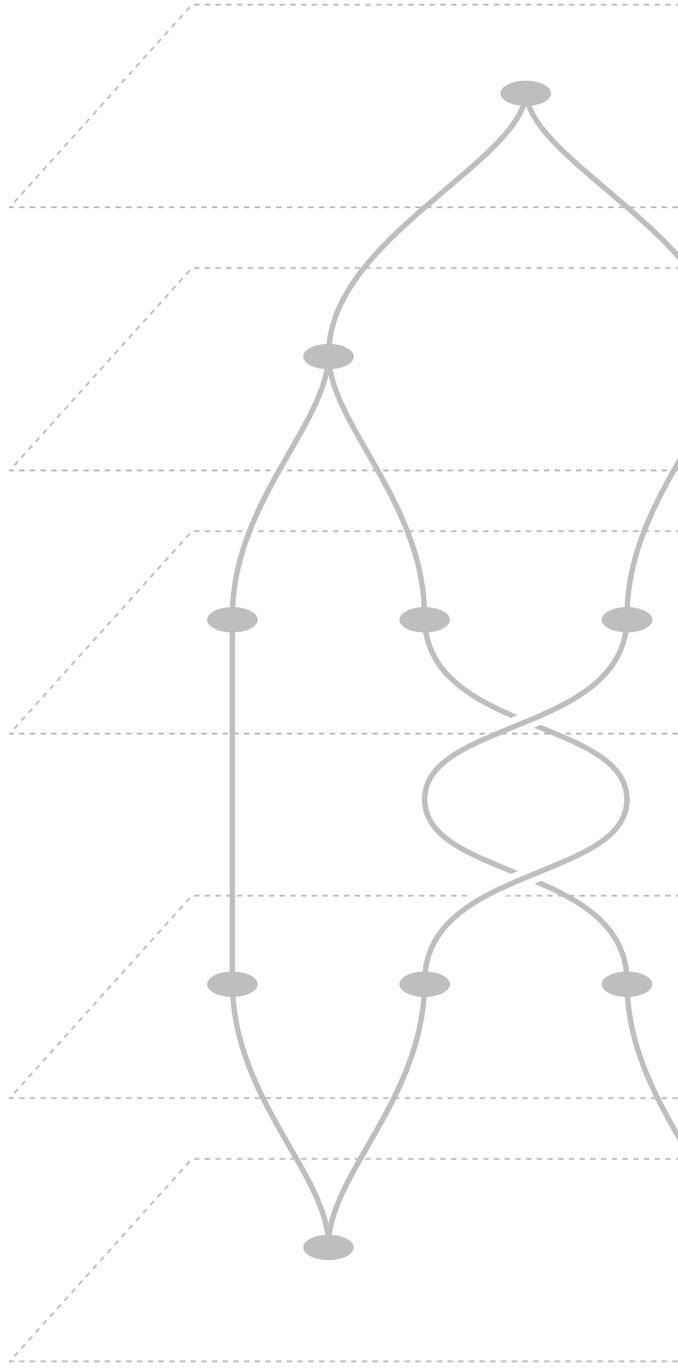
→ Robust **topological ground state degeneracies**

→ **Majorana chain**: 1D mean field superconductor

→ **Double chain**: Number-conserving analog of Majorana chain

→ **Ground states**: Equal-weight superpositions with fixed subchain parities *& fixed total fermion number*

→ **Robust degeneracy** except for **couplings at endpoints**



What is it good for?

→ Robust degeneracy realizes **qubits**

→ Coupling at endpoints useful for **braiding**

→ Braiding reveals **non-abelian Ising anyons**

→ Non-abelian anyons facilitate **robust gates** on qubits

→ **Topological Quantum Computation**