Towards topological quantum computation:
Minimal instances for toric code ground states

ITP III Institute Seminar

Nicolai Lang

Institute for Theoretical Physics III

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University of Stuttgart

Outline

1. **Stabilizer formalism & Graph states**
2. **The toric code model**
3. **LC classification of toric code ground states**
4. **Current projects**

Outline: Stabilizer formalism & Graph states

1. **Stabilizer formalism & Graph states**
   - The stabilizer formalism
   - A normal form for stabilizer states: Graph states
2. **The toric code model**
3. **LC classification of toric code ground states**
4. **Current projects**

The stabilizer formalism[1]

- The N-qubit Pauli group acting on $\mathcal{H}^N := \bigotimes_{i=1}^{N} \mathbb{C}_i^2$ is defined as

$$G_N := \operatorname{span}\left\{\mathbb{1}_1 \otimes \cdots \otimes \sigma_k^i \cdots \otimes \mathbb{1}_N \,\middle|\, i \in \{x, y, z\}; 1 \leq k \leq N, k \in \mathbb{N}\right\}$$

$$\hookrightarrow\ G_1 := \operatorname{span}\{\sigma^x, \sigma^y, \sigma^z\} = \{\pm\mathbb{1}, \pm i\mathbb{1}, \pm\sigma^x, \pm i\sigma^x, \pm\sigma^y, \pm i\sigma^y, \pm\sigma^z, \pm i\sigma^z\}$$

Definition: Stabilizer

Suppose that $\mathcal{G} = \{g_i\}_{i=1}^{d} \subseteq G_N$ is a given set of independent, pairwise commuting Pauli operators. Then

$$\mathcal{S} := \operatorname{span}\mathcal{G} \qquad \text{and} \qquad \mathcal{PS} := \left\{|\Phi\rangle \in \mathcal{H}^N \,\middle|\, \mathcal{S}\,|\Phi\rangle = |\Phi\rangle\right\}$$

are called the **stabilizer** $\mathcal{S}$ of the **protected subspace** $\mathcal{PS} \leq \mathcal{H}^N$.
$\mathcal{G}$ are the **generators** of $\mathcal{S}$ and $d = \operatorname{rank}\mathcal{S}$ is called the **rank** of the stabilizer.

- Note that $\mathcal{S}$ is an abelian subgroup of G_N by definition.
- If $-\mathbb{1} \notin \mathcal{S}$ it holds that $\dim\mathcal{PS} = 2^{N-d}$.

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[1] Nielsen and Chuang, *Quantum Computation and Quantum Information*.

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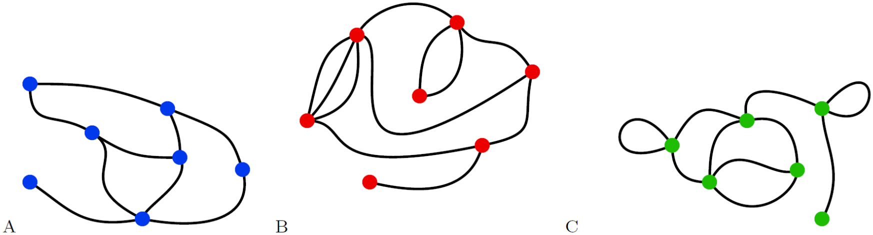
Graph theory: Simple graphs & Multigraphs[2]



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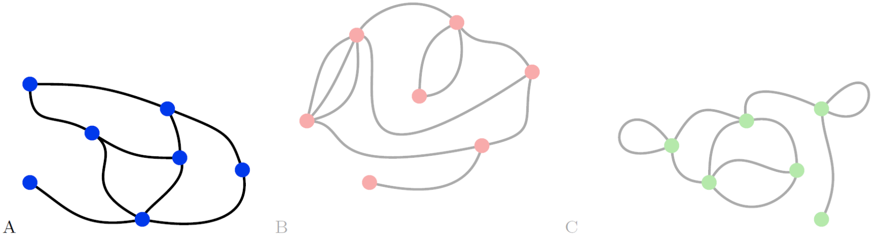

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Graph states: Definition[3]

- Consider simple graphs $G = (V, E)$ (no multiple edges, no loops).
- Write $\mathbb{V}(G) = V$ for the vertices and $\mathbb{E}(G) = E$ for the edges.
- For each vertex $v \in \mathbb{V}(G)$, N_v denotes the set of adjacent vertices.

Definition: Graph states

Let G be a simple graph. Consider a system of $N = |\mathbb{V}(G)|$ qubits, each identified with a vertex $v \in \mathbb{V}(G)$. Define the following (independent) operators

$$K_G^{(v)} := \sigma_v^x \prod_{w \in N_v} \sigma_w^z \in G_N \quad \text{and set} \quad \mathcal{G}[G] = \left\{ K_G^{(v)} \right\}_{v \in \mathbb{V}(G)}$$

Then $\mathcal{S}[G] = \operatorname{span} \mathcal{G}[G]$ is the **graph state stabilizer**. It is $\dim \mathcal{PS}[G] = 1$. The unique state $|G\rangle$ ($\mathcal{PS}[G] = \operatorname{span}\{|G\rangle\}$) is called the **graph state** described by G.

$\hookrightarrow$ Graph states are highly entangled multi-qubit states.

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Graph states: LC equivalence[4]

- The Clifford group $\mathcal{C}_N$ is defined as the normalizer of G_N, i.e.

$$\mathcal{C}_N := N_{U(2^N)}\left(G_N\right) = \left\{u \in U\left(2^N\right) \,\middle|\, uG_Nu^\dagger = G_N\right\}$$

$$\hookrightarrow \mathcal{C}_1 = \left\{u \in U(2) \,\middle|\, uG_1u^\dagger = G_1\right\}$$

- The local Clifford group $\mathcal{C}_N^l$ is defined as

$$\mathcal{C}_N^l := \{c_1 \otimes \cdots \otimes c_N \,|\, c_i \in \mathcal{C}_1, 1 \leq i \leq N\} = \mathcal{C}_1^{\otimes N}$$

↪ Clifford operations $C \in \mathcal{C}_N$ map stabilizer states onto stabilizer states.
↪ Local Clifford (LC) operations $C_l \in \mathcal{C}_N^l$ are local unitaries (LU).

Proposition: Stabilizer and graph states

Let $|\Psi\rangle$ be an arbitrary stabilizer state on N qubits. Then there exists a graph state $|G\rangle$ and a local Clifford operation $C \in \mathcal{C}_N^l$ such that $|\Psi\rangle = C\,|G\rangle$.

↪ Graph states are universal standard forms of stabilizer states.

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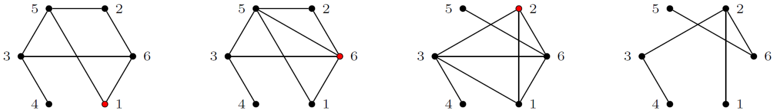
Graph states: The LC rule[5]

Definition: Local complementations

Let G be a simple graph and $v \in \mathbb{V}(G)$. $\langle N_v \rangle$ denotes the complete graph on the neighbourhood of v. Then $\tau_v(G) := G + \langle N_v \rangle$ is called the **local complement** of G at vertex v. If there is a sequence of **local complementations** $(\tau_i)_{i \in I}$ connecting two graphs G and G', we call them **LC equivalent** (write $G \sim_{\mathrm{LC}} G'$).



Proposition: LC rule

Let $|G\rangle$ and $|G'\rangle$ be two graph states with graphs $G = (E, V)$ and $G' = (E', V)$. Then it holds that

$$|G\rangle \sim_{\mathrm{LC}} |G'\rangle \Leftrightarrow G \sim_{\mathrm{LC}} G' .$$

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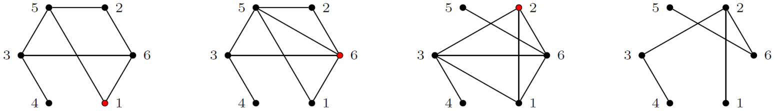

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Outline: The toric code model

1. Stabilizer formalism & Graph states

2. **The toric code model**
   - Topological quantum computation
   - A toy model
   - Implementations of the toric code

3. LC classification of toric code ground states

4. Current projects

Three models of quantum computation

- Quantum Circuit Model:
  ↪ "Classical" approach: Sequence of quantum gates on N-qubit register.
  - ▷ Michael A. Nielsen and Isaac L. Chuang. *Quantum Computation and Quantum Information*. 10th Anniversary. Cambridge University Press, Dec. 2010
- Measurement based Quantum Computation:
  ↪ Entanglement as a resource: Sequence of measurements on 2D cluster states (graph states) which are destroyed during the computation.
  - ▷ Robert Raussendorf, Daniel E. Browne, and Hans J. Briegel. "Measurement-based quantum computation on cluster states". In: *Phys. Rev. A* 68 (2 2003), p. 022312
- Topological Quantum Computation (TQC):
  ↪ Inherent fault tolerance: Creation, braiding and annihilatin of anyonic excitations performs unitary operations on the degenerate ground state space.
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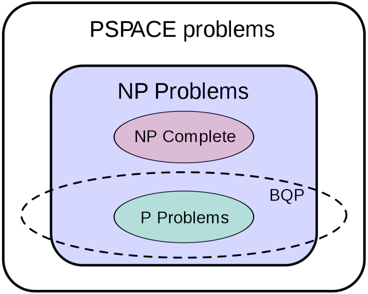
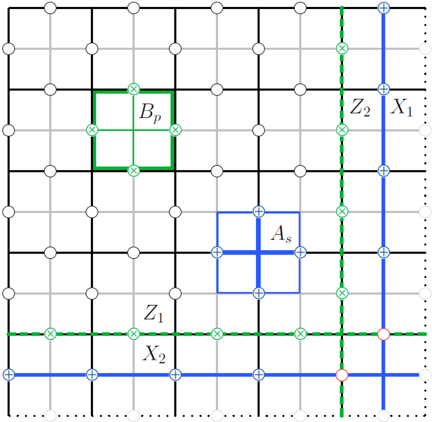
Three models of quantum computation

Note

All three models are computationally equivalent
(i.e. they give rise to the same complexity classes).



The toric code model (TCM): Definition[6]



- $\mathcal{L}$: square lattice with N cells in each direction.
- $\mathbb{P}(\mathcal{L})$: set of faces on $\mathcal{L}$.
- Embed the lattice $\mathcal{L}$ into the torus.
- Attach a single qubit to each of the $2N^2$ edges: $\mathcal{H} = \bigotimes_{e\in\mathbb{E}(\mathcal{L})} \mathbb{C}^2_e$.
- Define a stabilizer as follows ...

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[6] A.Yu. Kitaev. "Fault-tolerant quantum computation by anyons". In: *Annals of Physics* 303.1 (2003), pp. 2 –30.

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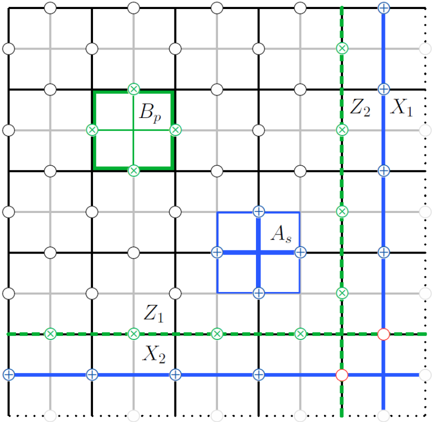

- Define star operators

$$A_s := \prod_{i \in s} \sigma_i^x \qquad s \in \mathbb{V}(\mathcal{L})$$

and plaquette operators

$$B_p := \prod_{i \in p} \sigma_i^z \qquad p \in \mathbb{P}(\mathcal{L})$$

- Obviously

$$[A_s, A_{s'}] = [B_p, B_{p'}] = [A_s, B_p] = 0$$

and $A_s, B_p \in G_{2N^2}$.

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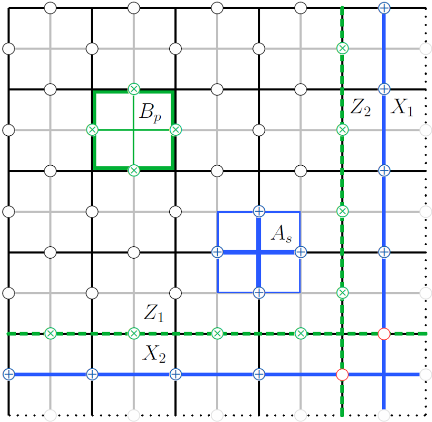

- The toric code stabilizer is

$$\mathcal{S} := \text{span}\,\{A_s, B_p\}_{s\in\mathbb{V}(\mathcal{L}),p\in\mathbb{P}(\mathcal{L})}$$

- rank $\mathcal{S}$ is not $2N^2$ since

$$\prod_{s\in\mathbb{V}(\mathcal{L})} A_s = \mathbb{1} = \prod_{p\in\mathbb{P}(\mathcal{L})} B_p$$

- Thus the protected space is degenerate:

$$\dim \mathcal{PS} = 2^{2N^2-(2N^2-2)} = 2^2 = 4$$

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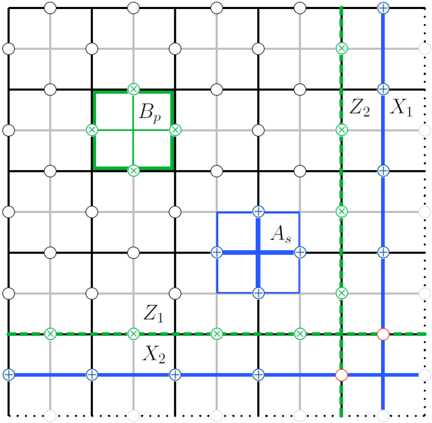

- The protected space constitutes a 2-qubit quantum code:
- Define string operators:

$$X\left[\mathcal{C}^*\right] := \prod_{i^*\in\mathcal{C}^*} \sigma^x_{i^*}\,, \qquad Z\left[\mathcal{C}\right] := \prod_{i\in\mathcal{C}} \sigma^z_i$$

- For $\mathcal{C}, \mathcal{C}^*$ closed loops:

$$\begin{aligned} \left[A_s, X\left[\mathcal{C}^*\right]\right] = \left[B_p, X\left[\mathcal{C}^*\right]\right] &= 0 \\ \left[A_s, Z\left[\mathcal{C}^*\right]\right] = \left[B_p, Z\left[\mathcal{C}^*\right]\right] &= 0 \end{aligned}$$

$\hookrightarrow$ Z,X diagonalizable over $\mathcal{PS}$.

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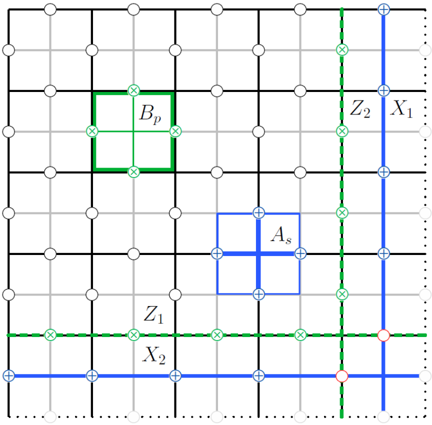

- Choose homologously non-trivial loops $\hookrightarrow X_1, X_2, Z_1, Z_2$.
- The corresponding loop operators obey the Pauli spin algebra:

$$[Z_i, X_j] = [X_i, X_j] = [Z_i, Z_j] = 0\,,$$

$$\{Z_i, X_i\} = 0\,, \quad i \neq j.$$

- Choose an eigenbasis of Z_1, Z_2 on $\mathcal{PS}$:

$$Z_j\,|v_1, v_2\rangle = v_j\,|v_1, v_2\rangle\;, \quad v_j \in \{-1, 1\}$$

- $\{|v_1, v_2\rangle\}_{v_i \in \{1,-1\}}$ is a ONB of $\mathcal{PS}$; v_j: topological quantum numbers.

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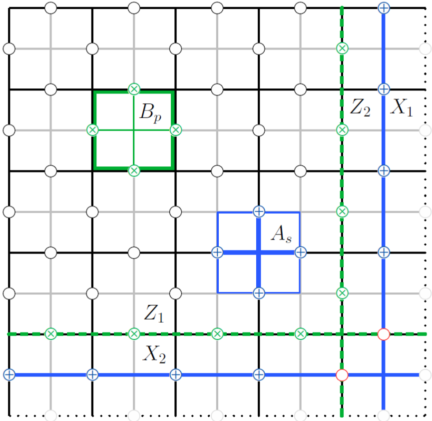

- $|v_1, v_2\rangle$ is a logical 2-qubit state.
- Flip qubits by X_j:

$$X_j\,|v_1, v_2\rangle = \begin{cases} |-v_1, v_2\rangle\,, & j = 1 \\ |v_1, -v_2\rangle\,, & j = 2 \end{cases}$$

- Measure qubits by Z_j.
  ↪ Encoding 2 logical qubits into $2N^2$ physical qubits.
- In general: $\mathcal{L}$ embedded into an orientable, compact 2-manifold with genus g ↪ $\dim \mathcal{PS} = 4^g$ ($2g$ qubits).

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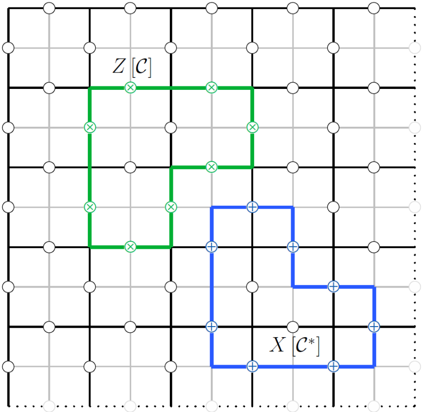
Trivial operations on $\mathcal{PS}$: Homology and the TCM



- Homology group: $[\mathcal{C}_1] = [\mathcal{C}_2] \in H_1$ iff $\mathcal{C}_1$ and $\mathcal{C}_2$ differ by a set of faces $p \in \mathbb{P}(\mathcal{L})$.
- $\mathcal{C}$ boundary of faces $\hookrightarrow$ $[\mathcal{C}] = 1 \in H_1$.
- Furthermore: $Z[\mathcal{C}]$ is the product of plaquette operators $\hookrightarrow$ $Z[\mathcal{C}] = \mathbb{1} \in \operatorname{Aut}\mathcal{PS}$.
- We conclude: $Z : H_1 \to \operatorname{Aut}\mathcal{PS}$ is a representation of H_1 on $\mathcal{PS}$:

$$Z[\mathcal{C}_1 + \mathcal{C}_2] = Z[\mathcal{C}_1] \circ Z[\mathcal{C}_2]$$

- The same holds for X and H_1^*.

Elementary excitations and anyonic statistics

- *Up to now*: Toric Code Model = Quantum code

Definition: Toric Code Hamiltonian

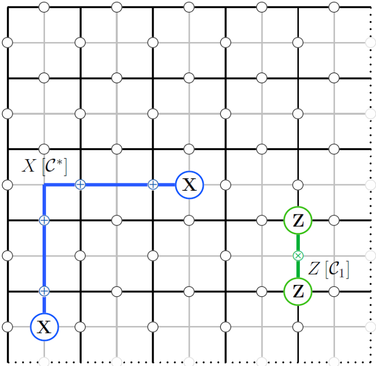
Let A_s and B_p be the star and plaquette operators defined previously. Then

$$H_{\mathrm{TCM}} := -J_A \sum_{s \in \mathbb{V}(\mathcal{L})} A_s - J_B \sum_{p \in \mathbb{P}(\mathcal{L})} B_p$$

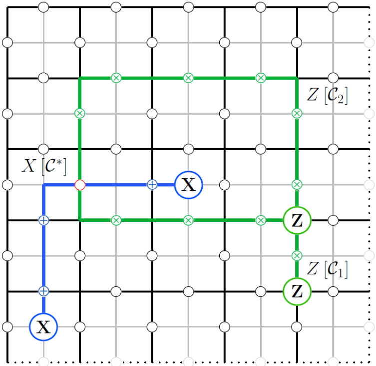
is a **quasilocal** Hamiltonian with **ground state space** $\mathcal{PS}$.

- *Now*: Toric Code Model = Physical model
- Elementary excitations:
  $A_s \left|\Psi\right\rangle = -\left|\Psi\right\rangle \hookrightarrow$ Z-type particle on s
  $B_p \left|\Psi\right\rangle = -\left|\Psi\right\rangle \hookrightarrow$ X-type particle on p
- Elementary excitations can only be generated in pairs.

Elementary excitations and anyonic statistics

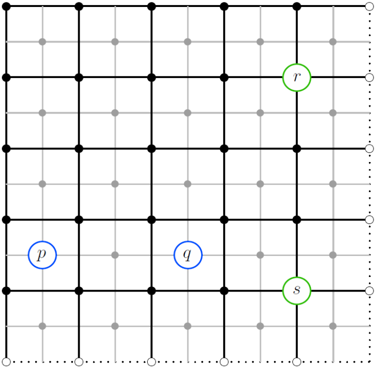


- Create X-type particle pair: $X[\mathcal{C}^*]$.
- Create Z-type particle pair: $Z[\mathcal{C}_1]$.

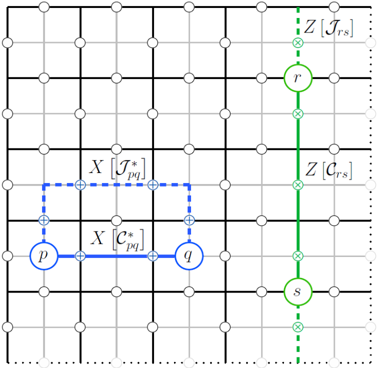


- Wind Z-type particle around X-type particle: $Z[\mathcal{C}_2]$.
- Since $\{Z[\mathcal{C}_2], X[\mathcal{C}^*]\} = 0 \hookrightarrow$ Phase $e^{i\pi} \hookrightarrow$ Mutual anyonic statistics.

Algorithmic quantum error correction



- Measure all star and plaquette operators
  $\hookrightarrow$ -1 Eigenvalues $\Leftrightarrow$ Excitation
- This is called the Error syndrome.



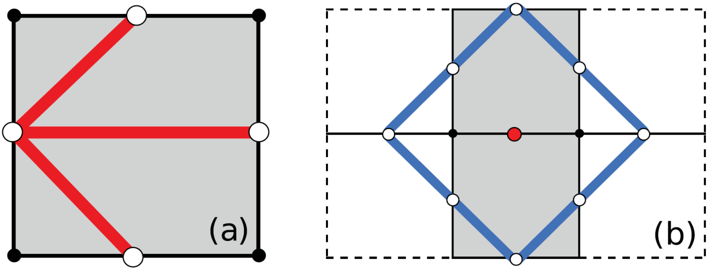
- Apply (shortest) string operators connecting pairs of excitations.
- Logical qubits recovered if the number of errors is at most $\lfloor \frac{N-1}{2} \rfloor$.

Implementations of the toric code

- Photonic quantum simulators (↪ Experiments)
  - Chao-Yang Lu, Wei-Bo Gao, Otfried Gühne, Xiao-Qi Zhou, Zeng-Bing Chen, and Jian-Wei Pan. “Demonstrating Anyonic Fractional Statistics with a Six-Qubit Quantum Simulator”. In: *Phys. Rev. Lett.* 102 (3 2009), p. 030502
  - J. K. Pachos, W. Wieczorek, C. Schmid, N. Kiesel, R. Pohlner, and H. Weinfurter. “Revealing anyonic features in a toric code quantum simulation”. In: *New Journal of Physics* 11.8 (Aug. 2009), p. 083010
- Rydberg quantum simulators (↪ Proposals)
  - H. Weimer, M. Müller, I. Lesanovsky, P. Zoller, and H. P. Büchler. “A Rydberg quantum simulator”. In: *Nature Physics* 6 (May 2010), pp. 382–388
- Ultracold atoms in optical lattices (↪ Proposals)
  - M. Aguado, G. K. Brennen, F. Verstraete, and J. I. Cirac. “Creation, Manipulation, and Detection of Abelian and Non-Abelian Anyons in Optical Lattices”. In: *Phys. Rev. Lett.* 101 (26 2008), p. 260501
  - Belén Paredes and Immanuel Bloch. “Minimum instances of topological matter in an optical plaquette”. In: *Phys. Rev. A* 77 (2 2008), p. 023603
  - L. Jiang, G. K. Brennen, A. V. Gorshkov, K. Hammerer, M. Hafezi, E. Demler, M. D. Lukin, and P. Zoller. “Anyonic interferometry and protected memories in atomic spin lattices”. In: *Nature Physics* 4 (Apr. 2008), pp. 482–488

Outline: LC classification of toric code ground states

Introducing the notion of "locality"[7]



- TCM ground states = stabilizer states $\hookrightarrow \exists$ LC equivalent graph state $|G\rangle$.
- Example above: Single plaquette $\sim$ Star graph $\sim$ GHZ state.

- A_s and B_p give rise to (physically motivated) adjacency relation: q_1, q_2 adjacent $:\Leftrightarrow \exists$ common A_s and/or B_p
- On the square lattice: 8 neighbours.

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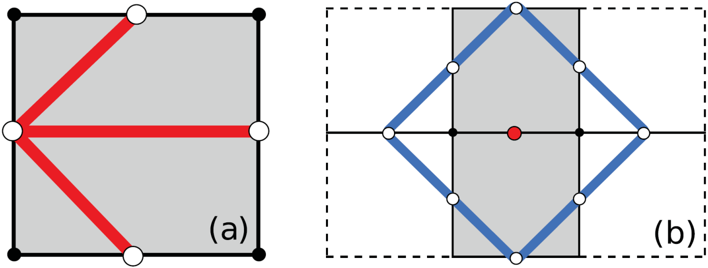
[7] Nicolai Lang and Hans Peter Büchler. "Minimal instances for toric code ground states". In: *Phys. Rev. A* 86 (2 2012), p. 022336.

Introducing the notion of "locality"[7]



Definition: Local graph states & TCM setups

Local graph state $\equiv$ Edges connect only adjacent qubits.
Local TCM setup $\equiv$ There exists at least one LC equivalent, local graph state.

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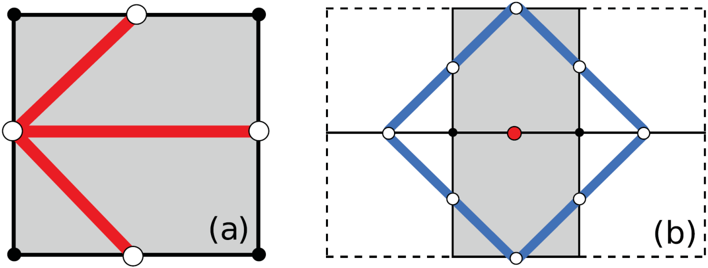
[7] Nicolai Lang and Hans Peter Büchler. "Minimal instances for toric code ground states". In: *Phys. Rev. A* 86 (2 2012), p. 022336.

Introducing the notion of “locality”[7]



Question: Minimal non-local setup?

Obviously small TCM setups are local (see single plaquette above).
Which is the smallest non-local TCM setup (on the square/triangular lattice)?

↪ Efficient method to transform TCM setups into LC equivalent graph states?

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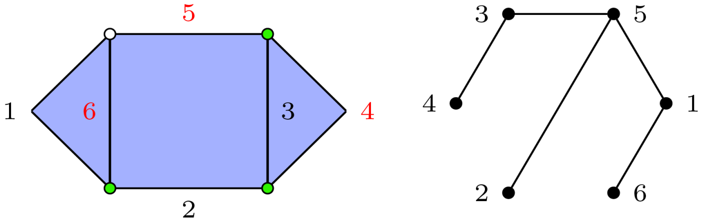
[7] Nicolai Lang and Hans Peter Büchler. “Minimal instances for toric code ground states”. In: *Phys. Rev. A* 86 (2 2012), p. 022336.

Example: LC transformation

- Consider the 6-qubit TCM setup on the left side.
- An LC equivalent graph state is depicted on the right side.



- How did we obtain the graph state?
  ↪ Cumbersome method: Compute it with an CAS.
  ↪ Neat method: Just *draw* it!

A (cumbersome) algorithmic approach[8]

1. Encode the TCM stabilizer into binary a matrix.
2. Choose an appropriate order of columns.
3. Choose an appropriate order of rows.
4. Apply Hadamard transformations via $\mathbf{H}$.
5. Apply a basis change via $\mathbf{T}''$.
6. Extract the adjacency matrix $\mathbf{\Gamma}_1 \hookrightarrow$ Graph state $|G\rangle$.

$$\mathbf{M}_1 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{where} \quad \mathbf{R}_x = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{R}_x^1 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \in \mathrm{GL}\left(3, \mathbb{F}_2\right)$$

---

[8] Maarten Van den Nest, Jeroen Dehaene, and Bart De Moor. "Graphical description of the action of local Clifford transformations on graph states". In: *Phys. Rev. A* 69 (2 2004), p. 022316.

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[8] Maarten Van den Nest, Jeroen Dehaene, and Bart De Moor. “Graphical description of the action of local Clifford transformations on graph states”. In: *Phys. Rev. A* 69 (2 2004), p. 022316.

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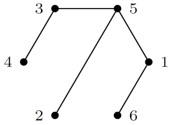

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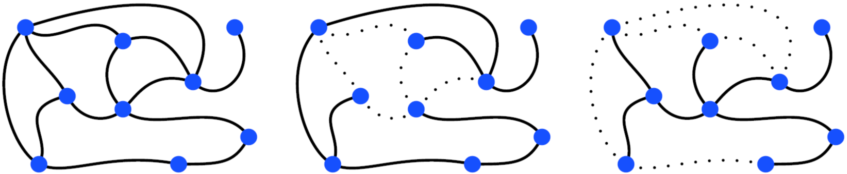
[8] Maarten Van den Nest, Jeroen Dehaene, and Bart De Moor. "Graphical description of the action of local Clifford transformations on graph states". In: *Phys. Rev. A* 69 (2 2004), p. 022316.

Graph theory: Spanning trees $\mathcal{L}'$

- Consider multigraphs $\mathcal{L}$.
- Spanning tree $\mathcal{L}'$: Connected, loopless subgraph which reaches all vertices.
- Construction: Delete successively all edges that support loops.
- Example:



$\varphi_{\mathcal{L}'}$ and an (elegant) graph theoretic approach[9]

Definition: Graph transformation $\varphi_{\mathcal{L}'}(\mathcal{L})$

Let $\mathcal{L}$ be a multigraph and $\mathcal{L}'$ one of its spanning trees. Then $G = \varphi_{\mathcal{L}'}(\mathcal{L})$ is a simple graph defined by the following construction:

Set $\mathbb{V}(G) := \mathbb{E}(\mathcal{L})$ as the vertex set of G. For $r, s \in \mathbb{V}(G)$ the set $\{r, s\}$ is an edge of G if and only if there is a path $\mathcal{C}_{pq}$ from p to q on $\mathcal{L}'$ such that $r = \{p, q\}$ and $s \in \mathcal{C}_{pq}$.

↪ Example: Blackboard...

Theorem: Transformation rule

Given the TCM ground state $|\mathbf{v}\rangle$ on $\mathcal{L}$ and any spanning tree $\mathcal{L}'$. Then the graph state $|G\rangle = |\varphi_{\mathcal{L}'}(\mathcal{L})\rangle$ is LC equivalent to $|\mathbf{v}\rangle$.

↪ Corollary: For arbitrary spanning trees $\mathcal{L}'_1$, $\mathcal{L}'_2$ it holds $\varphi_{\mathcal{L}'_1}(\mathcal{L}) \sim_{\mathrm{LC}} \varphi_{\mathcal{L}'_2}(\mathcal{L})$.

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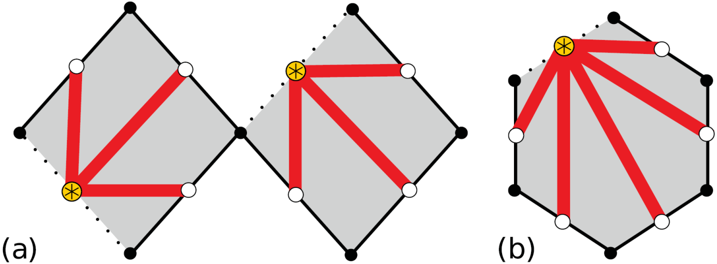
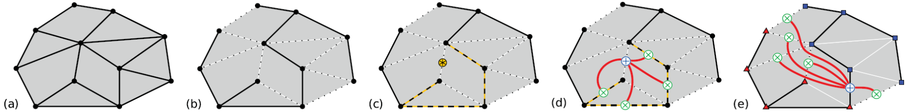
Examples



(a) One-point connected TCM setups
∼ disconnected graph states
∼ product states ✓

(b) Single TCM plaquettes
∼ star graphs
∼ GHZ states ✓

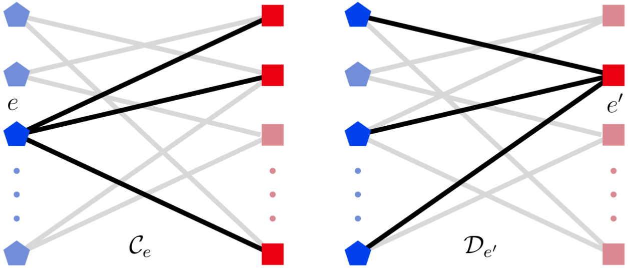
Transformation rule: Proof



- Poof via the stabilizer framework.
- Show that $\mathcal{S}$ becomes $\mathcal{S}[\varphi_{\mathcal{L}'}(\mathcal{L})]$ via Hadamard operations on "deleted" edges.
- For details see
  - ▷ Nicolai Lang. "Minimal instance for topological matter". B.Sc. Thesis. University of Stuttgart, 2011
  - ▷ Nicolai Lang and Hans Peter Büchler. "Minimal instances for toric code ground states". In: *Phys. Rev. A* 86 (2 2012), p. 022336

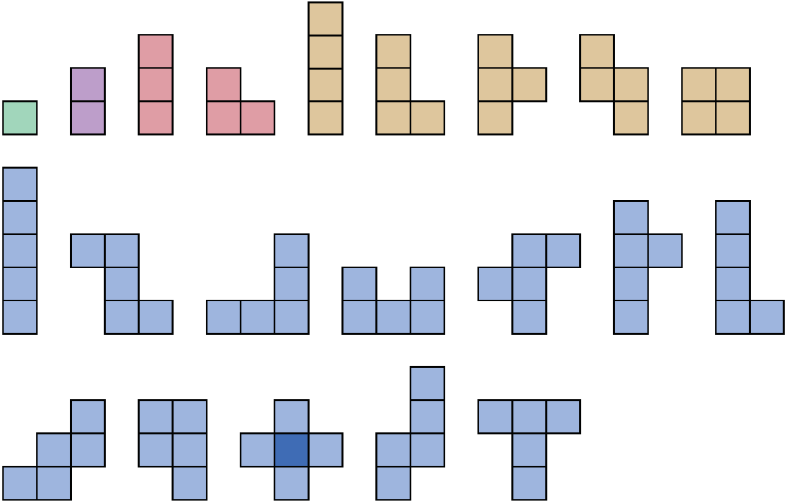
A graph theoretic interpretation of $\varphi_{\mathcal{L}'}(\mathcal{L})$

- Cycle space and cut space: linear subspaces of the binary edge space.
- Basis for the cycle space: fundamental cycles $\mathcal{C}_e$.
- Basis for the cut space: fundamental cuts $\mathcal{D}_{e'}$.

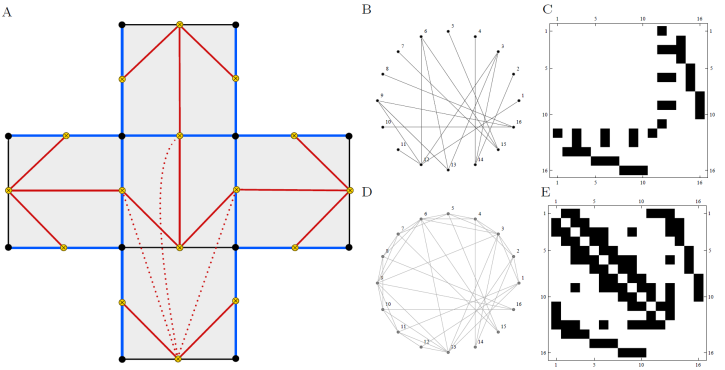


- Deleted edges e encode fundamental cycles $\mathcal{C}_e$ via $N_e \cup \{e\}$.
- Spanning tree edges e' encode fundamental cuts $\mathcal{D}_{e'}$ via $N_{e'} \cup \{e'\}$.

An enumeration of the smallest TCM setups



Minimal non-local candidate on the square lattice



(A) Spanning tree: blue.
Local edges: red (solid).
Non-local edges: red (dotted).

(C) Adjacency matrix of the graph state.

(E) Adjacency matrix encoding the physical vicinity of qubits.

A computational proof

- Problem: Not every element in $[\varphi_{\mathcal{L}'}(\mathcal{L})]_{\mathrm{LC}}$ is reached by $\varphi_{\cdot}(\mathcal{L})$.
- Goal: Check if the LC class of the candidate contains only non-local graphs.
  ↪ Generate the whole LC class $[G]_{\mathrm{LC}}$ for a given graph G.

Proposition: LC equivalent graphs[10]

Consider two graphs G and G' with adjacency matrices $\mathbf{\Gamma}$ and $\mathbf{\Gamma}'$. Then $G \sim_{\mathrm{LC}} G'$

$$\Leftrightarrow \exists\, \mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D} \in \mathbb{F}_2^{N\times N} \,:\, \mathbf{AD} + \mathbf{BC} = \mathbf{E}_N \wedge (\mathbf{\Gamma B} + \mathbf{D})\,\mathbf{\Gamma}' + (\mathbf{\Gamma A} + \mathbf{C}) = \mathbf{0}$$

- Using this proposition and `MATHEMATICA`, $[G]_{\mathrm{LC}}$ can be generated.
- Result: $[G]_{\mathrm{LC}}$ contains no local graphs.

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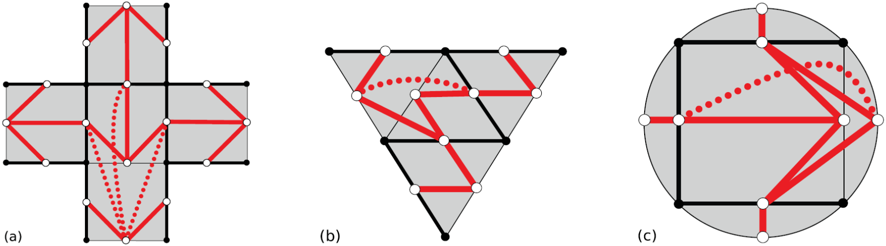
[10] André Bouchet. “An efficient algorithm to recognize locally equivalent graphs”. In: *Combinatorica* 11 (4 1991), pp. 315–329.

Conclusion: Results for square/triangular lattice



(a) Square lattice:
5 plaquettes, 16 qubits.

(b) Triangular lattice:
4 plaquettes, 9 qubits.

(c) Used to prove the non-locality of (a) (↪ reduction methods).

Outline: Current projects

1. **Stabilizer formalism & Graph states**
2. **The toric code model**
3. **LC classification of toric code ground states**
4. **Current projects**
   - Protecting topological order by dissipation
   - The Ising model with transverse magnetic field

1. Dissipatively cooling the majorana chain

- Consider a 1D fermionic chain (OBC) with the Hamiltonian[11]

$$H = \sum_j \left[-w\left(a_j^\dagger a_{j+1} + a_{j+1}^\dagger a_j\right) - \mu\left(a_j^\dagger a_j - \frac{1}{2}\right) + \Delta a_j a_{j+1} + \Delta^* a_{j+1}^\dagger a_j^\dagger \right]$$

- There exist unpaired Majorana fermions (boundary modes)
  ↪ store logical qubits.
- These obey non-abelian anyonic statistics in wire networks.
- Excitations: Bogoliubov particles ↪ destroy logical qubits.

Question: Dissipative cooling possible?

Is there a dissipative cooling mechanism[12] which removes the Bogoliubov particles while preserving the topological order?

---

[11] A.Yu. Kitaev. "Unpaired Majorana fermions in quantum wires". In: *Physics-Uspekhi* 131 (2007).

[12] Sebastian Diehl, Enrique Rico, Mikhail a. Baranov, and Peter Zoller. "Topology by dissipation in atomic quantum wires". In: *Nature Physics* 7.12 (Oct. 2011), pp. 971–977.

2. Dissipatively driven phasetransitions

- Consider the Ising model with transverse magnetic field[13]

$$H = -J \sum_{\langle n,m \rangle} \sigma_n^z \sigma_m^z - h \sum_n \sigma_n^x$$

- There is a quantum phase transition at finite $\frac{h}{J}$ (ferromagnetic ⇔ paramagnetic).
- Introduce jump operators c_j and d_j driving the system towards the paramagnetic and ferromagnetic ground state, respectively.
- Dynamics described by Lindblad master equation:

$$\partial_t \rho = -i[H, \rho] + \kappa_P \mathcal{L}(c_j)[\rho] + \kappa_F \mathcal{L}(d_j)[\rho]$$

Question: Stationary states in dependence of system parameters?

What are the properties of the stationary states in dependence of κ_P and κ_F? How does the phase diagram look like?

---

[13] Subir Sachdev. *Quantum Phase Transitions*. 2nd ed. Cambridge University Press, May 2011.

That's it!

Thank you for your attention.