Majorana Modes in an interacting & number-conserving Theory

Nicolai Lang
Institute for Theoretical Physics III
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CO.CO.MAT
Control of Quantum Correlations in Tailored Matter
SFB/TRR 21 - Stuttgart, Ulm, Tübingen

Stuttgart-Tübinger Doktorandenseminar
23. Januar 2015

Spectral Theory and Dynamics of Quantum Systems
GRADUIERTENKOLLEG 1838

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Where are we?

М1
УЛИЦА ПОДБЕЛЬСКОГО
ЧЕРКИЗОВСКАЯ
ПРЕОБРАЖЕНСКАЯ ПЛОЩАДЬ
СОКОЛЬНИКИ
КРАСНОСЕЛЬСКАЯ
КОМСОМОЛЬСКАЯ
КРАСНЫЕ ВОРОТА
ЧИСТЫЕ ПРУДЫ
ТУРГЕНЕВСКАЯ
КУЗНЕЦКИЙ МОСТ
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ОХОТНЫЙ РЯД
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ИЗМАЙЛОВСКАЯ
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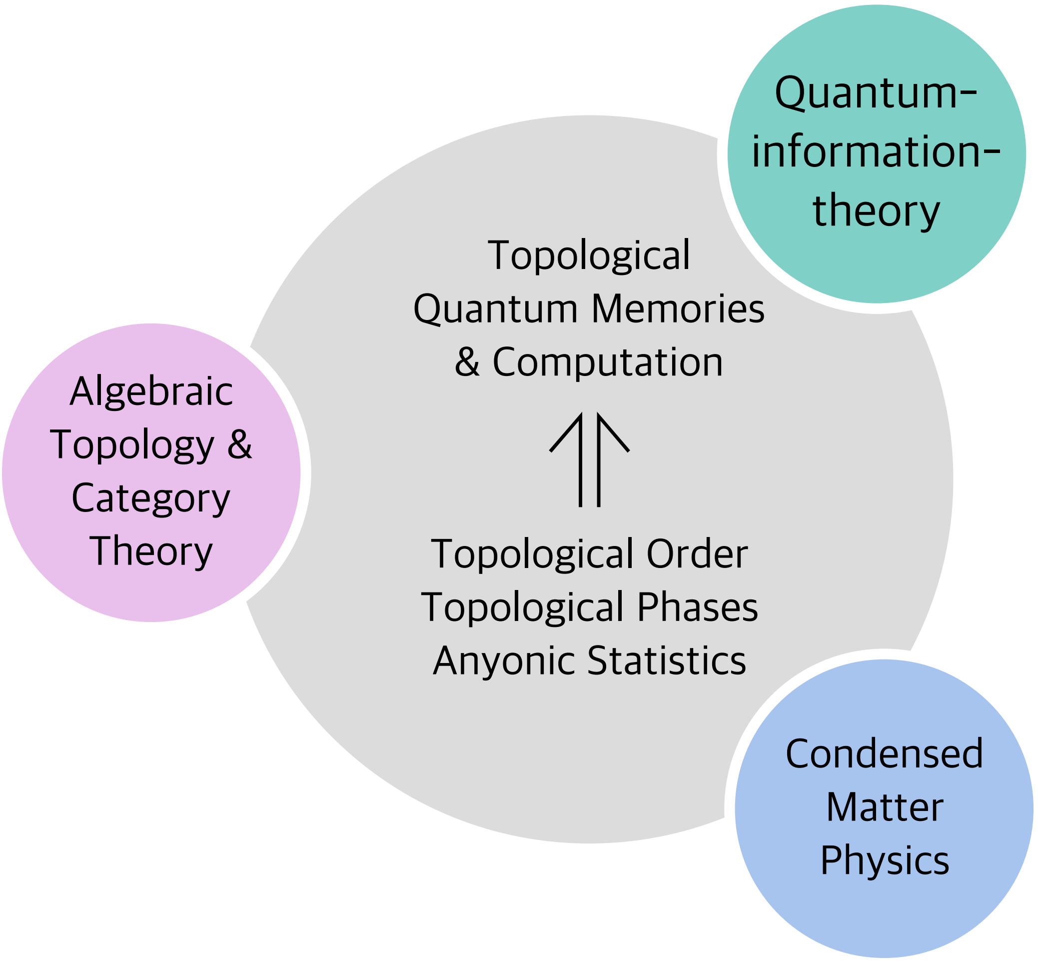
СЕМЕНОВСКАЯ
ЭЛЕКТРОЗАВОДСКАЯ


Quantum-
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Topological
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& Computation
Algebraic
Topology &
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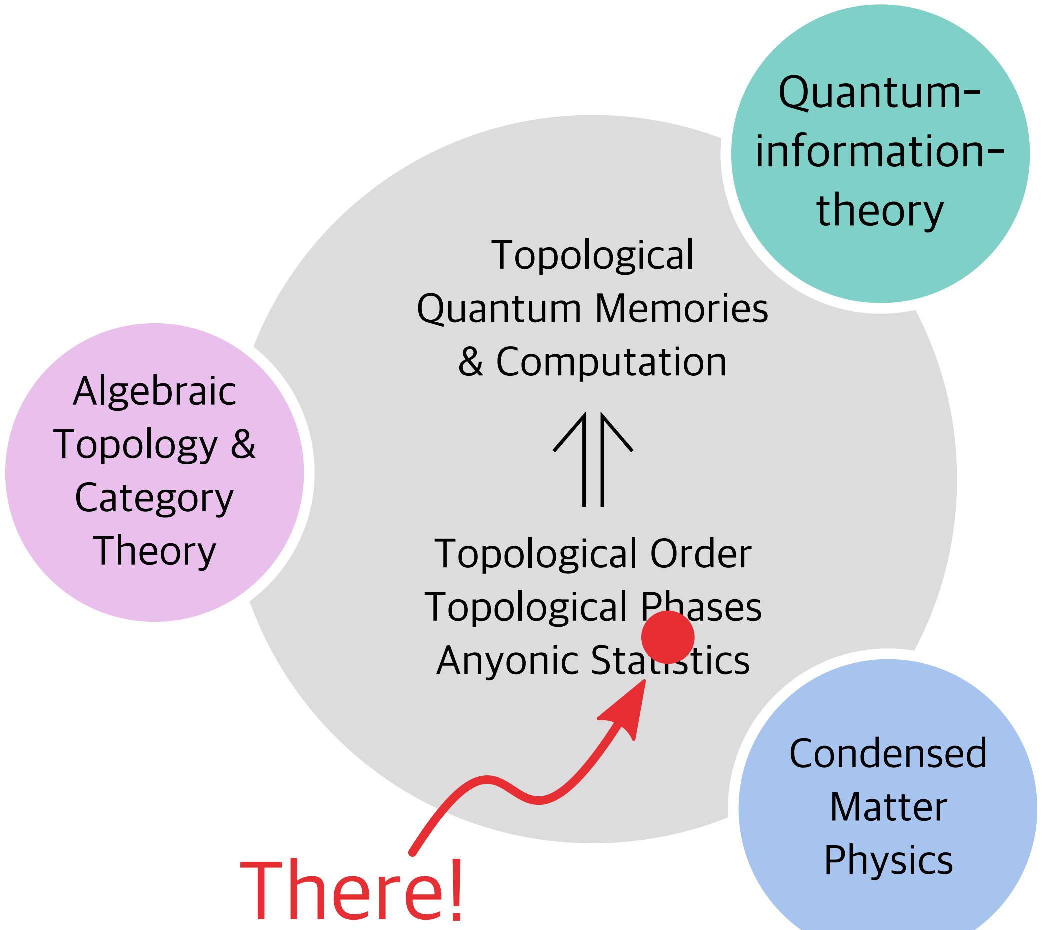
Theory
Topological Order
Topological Phases
Anyonic Statistics
Condensed
Matter
Physics


Quantum-information-theory
Topological Quantum Memories & Computation
Algebraic Topology & Category Theory
Topological Order
Topological Phases
Anyonic Statistics
Condensed Matter Physics
There!

Motivation

Kitaev's Majorana Chain

1 Motivation

$$\{a_i, a_j\} = 0 = \left\{a_i^\dagger, a_j^\dagger\right\} \quad \text{and} \quad \left\{a_i, a_j^\dagger\right\} = \delta_{i,j}$$

Paradigmatic Model:

Open chain of spinless fermions ↪ Kitaev Chain

$$H = -\sum_{i=1}^{L-1}\left[w a_i^\dagger a_{i+1} - |\Delta| a_i a_{i+1} + \text{h.c.}\right] - \mu \sum_{i=1}^{L}\left(a_i^\dagger a_i - \frac{1}{2}\right)$$

Majorana Fermions:

"Real & Imaginary part of fermionic generators"

$$c_{2i-1} \equiv a_i + a_i^\dagger \quad \text{and} \quad c_{2i} \equiv i\left(a_i^\dagger - a_i\right) \quad \text{for} \quad i = 1, \ldots, L$$

⇒ Self-adjoint fermions:

$$\{c_l, c_m\} = 2\delta_{l,m} \qquad \text{and} \qquad c_l = c_l^\dagger$$

Kitaev, A. Unpaired Majorana fermions in quantum wires. Physics-Uspekhi 131 (2007)

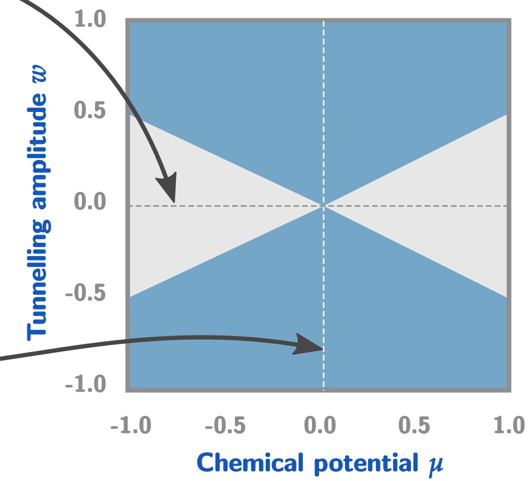
1 Motivation

$$a_i = \frac{1}{2}\left(c_{2i-1} + ic_{2i}\right) \quad \text{and} \quad a_i^\dagger = \frac{1}{2}\left(c_{2i-1} - ic_{2i}\right)$$

Trivial phase:

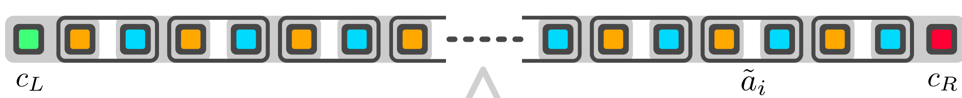
Dominant chemical potential

$$H = -\mu \frac{i}{2} \sum_{i=1}^{L} c_{2i-1} c_{2i}$$



Topological phase:

Dominant hopping & pairing



$$H = iw \sum_{i=1}^{L-1} c_{2i} c_{2i+1}$$

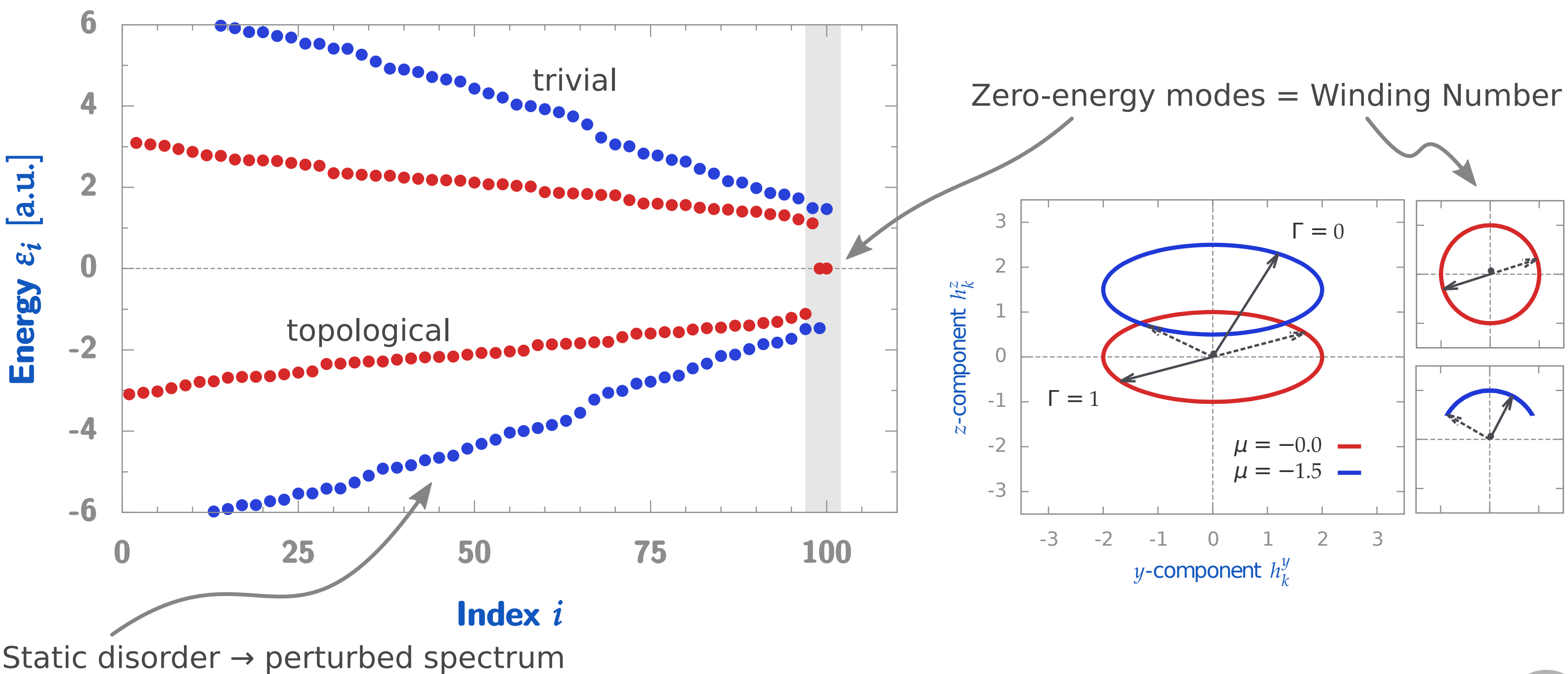
$$\tilde{a}_i \equiv \frac{1}{2}\left(c_{2i} + ic_{2i+1}\right) \quad \text{and} \quad \tilde{a}_i^\dagger \equiv \frac{1}{2}\left(c_{2i} - ic_{2i+1}\right)$$

⇒ Zero-energy edge mode

1 Motivation

Spectrum of Kitaev Chain:

Stable zero-energy modes in the topological phase:

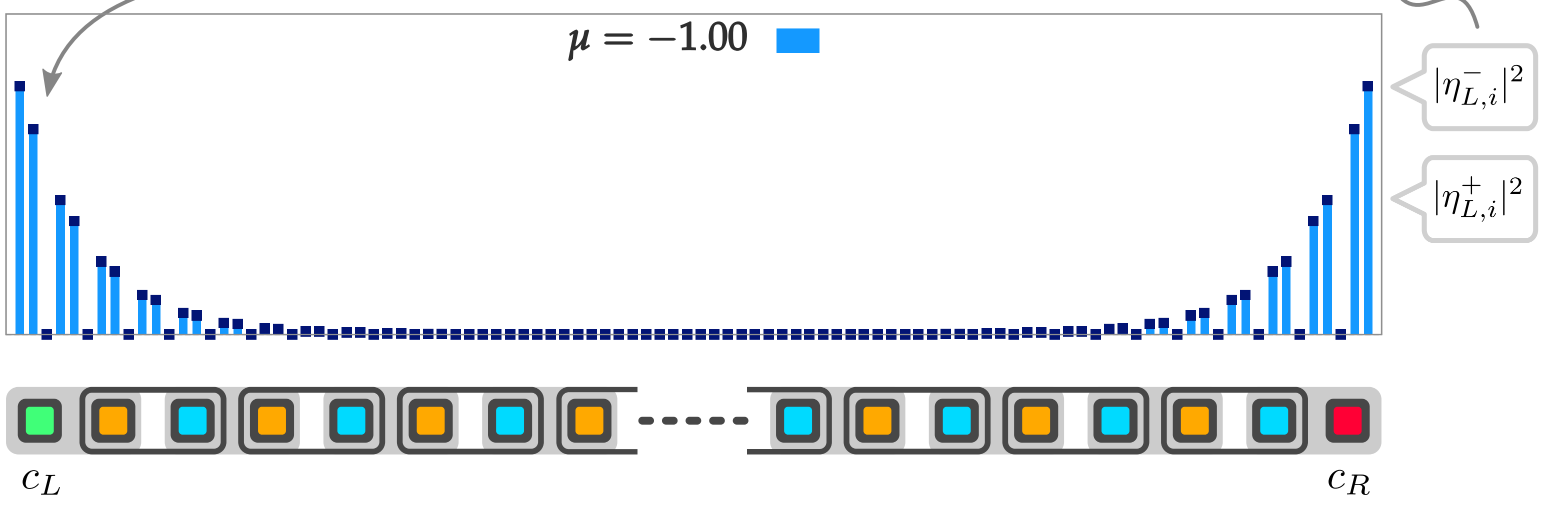


1 Motivation

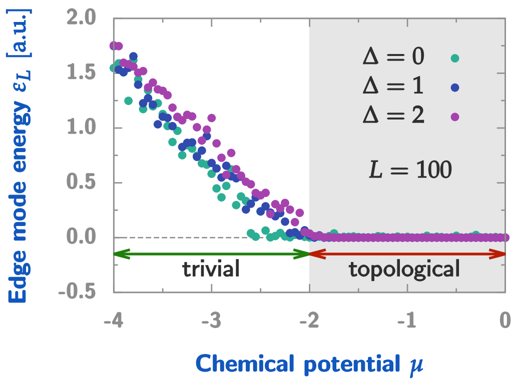
Wave functions:

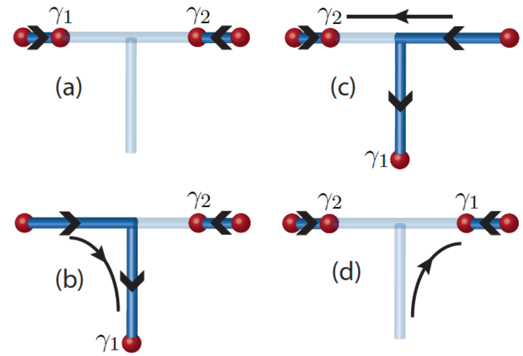
Zero-energy modes = Edge modes

$$\tilde{a}_L^\dagger = \sum_i \left(\eta_{L,i}^- a_i + \eta_{L,i}^+ a_i^\dagger \right)$$



① Why should we care?





Topological Phase:
Topological invariant protects ground state degeneracy

&

Anyonic Statistics:
Localized edge modes obey non-abelian anyonic statistics

↓

Inherently robust way of
storing & manipulating quantum information

⇒ Topological quantum memory & computer

Beyond Mean Field?

$$H = -\sum_{i=1}^{L-1}\left[w a_i^\dagger a_{i+1} - |\Delta| a_i a_{i+1} + \text{h.c.}\right] - \mu \sum_{i=1}^{L}\left(a_i^\dagger a_i - \frac{1}{2}\right)$$

Beyond Mean Field?

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… is not particle-number conserving.

Beyond Mean Field?

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A particle-number conserving Theory?

Beyond Mean Field?

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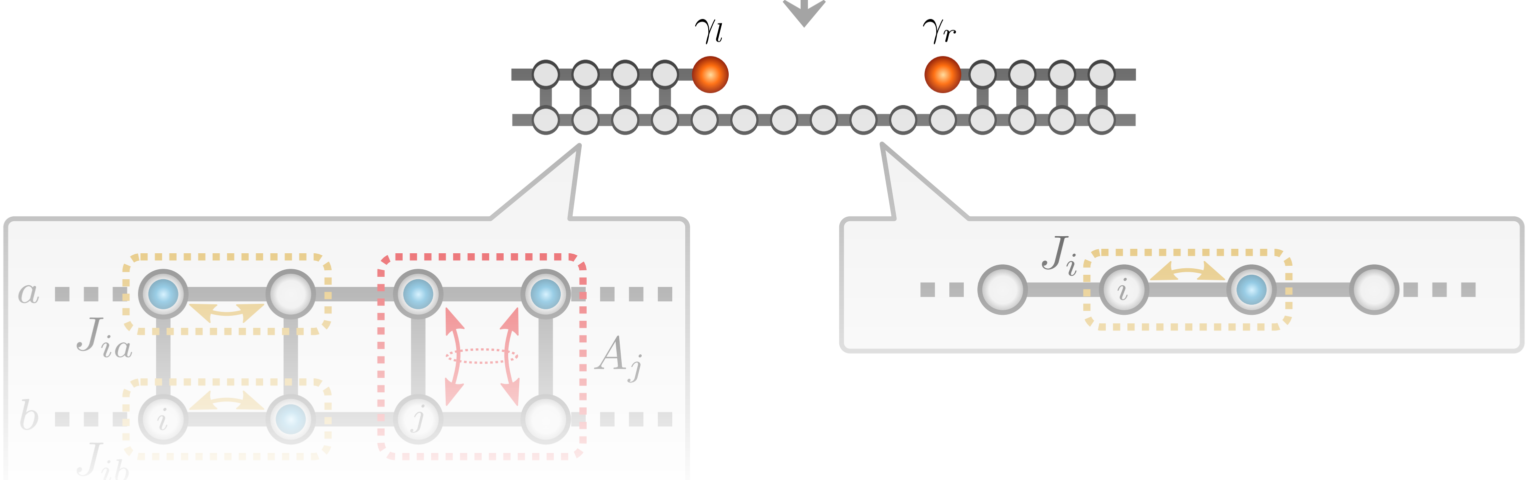
A particle-number conserving Theory?

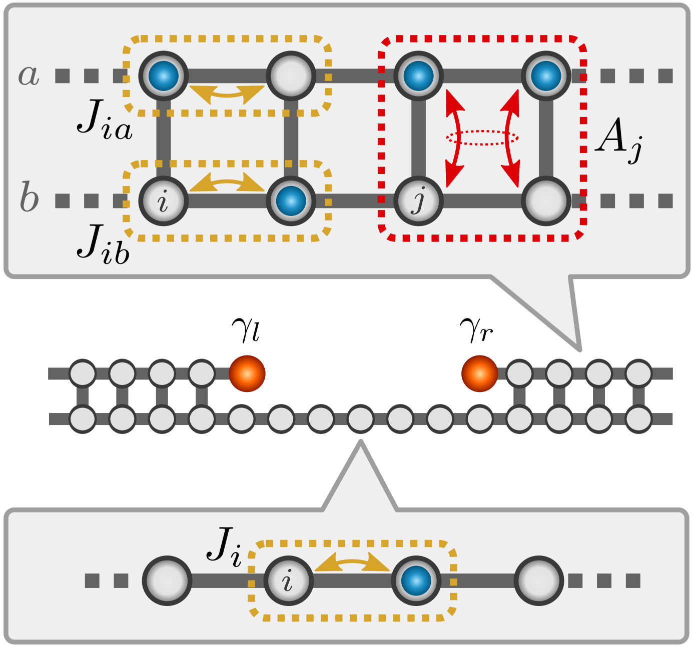
↓



② Beyond Mean Field?

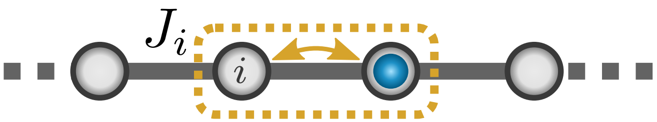
Our Model:
→ Double Chain
→ Spinless Fermions on Sites
→ Intra- & Interchain Interactions

$$H = \sum_{i=1}^{L} \left(H_i^a + H_i^b + H_i^{ab} \right)$$



Symmetries:
→ Particle-number
→ Time-reversal

② Beyond Mean Field?



Intrachain Interactions:

Single-particle hopping & NN density-density interactions

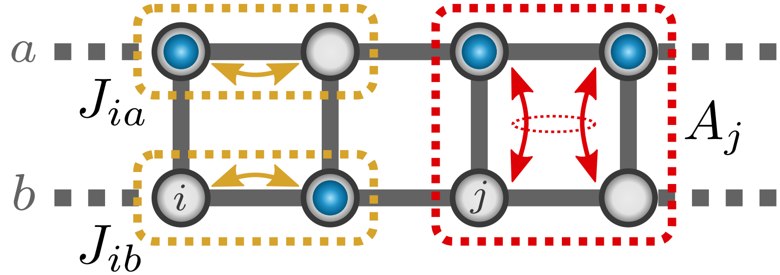
$$J_{ix} = \frac{1}{\sqrt{2}} \left(x_i^\dagger + x_{i+1}^\dagger \right) \left(x_i - x_{i+1} \right) \quad \text{for} \quad x = a, b$$

$$H_x = \sum_{i=1}^{L-1} J_{ix}^\dagger J_{ix}$$

Expanded form:

$$H_i^x = x_i x_{i+1}^\dagger + x_{i+1} x_i^\dagger + n_i^x (\mathbb{1} - n_{i+1}^x) + n_{i+1}^x (\mathbb{1} - n_i^x)$$

② Beyond Mean Field?



Interchain Interaction:

Pair-hopping & Pair-density-density interactions

$$A_i = a_i^\dagger a_{i+1}^\dagger b_i b_{i+1} + b_i^\dagger b_{i+1}^\dagger a_i a_{i+1}$$

$$H_{\text{int}} = \sum_{i=1}^{L-1} A_i \left(\mathbb{1} + A_i\right)$$

Expanded form:

$$H_i^{ab} = a_i^\dagger a_{i+1}^\dagger b_i b_{i+1} + b_i^\dagger b_{i+1}^\dagger a_i a_{i+1} + n_i^a n_{i+1}^a (\mathbb{1} - n_i^b)(\mathbb{1} - n_{i+1}^b) + n_i^b n_{i+1}^b (\mathbb{1} - n_i^a)(\mathbb{1} - n_{i+1}^a)$$

3 Exact Ground State

$$H_x = \sum_{i=1}^{L-1} J_{ix}^\dagger J_{ix} \qquad H_{\text{int}} = \sum_{i=1}^{L-1} A_i \left(\mathbb{1} + A_i\right)$$

Facts:

→ Each term in the Hamiltonian is a positive operator

→ Sum of positive operators = positive operator

Derivation of Exact Ground State (GS):

→ If zero-energy ⇒ GS = simultaneous kernel of local terms

→ If kernel trivial ⇒ Finite-energy GS (not accessible)

③ Exact Ground State

Do the math …

GS = Equal-weight superposition with fixed
total particle number & subchain parity

$$|N;\alpha\rangle = \mathcal{N} \sum_{M,\,(-1)^M=\alpha} |M\rangle_a \, |N-M\rangle_b$$

with

$$|M\rangle_x = \sum_{\boldsymbol{m}\in\{0,1\}^L,\,|\boldsymbol{m}|=M} |\boldsymbol{m}\rangle_x$$

Degeneracy for open boundary conditions:

Two GS for each particle number sector:

N even		
	even	odd
	even	odd

N odd		
	even	odd
	odd	even

Note:

Does not work for periodic boundary conditions in all sectors!

4 GS Properties

Evaluation of arbitrary correlators:

Pure Combinatorics!

Generalization:

Parity-split Binomial Coefficients (PBC)

$$\binom{L_1, \ldots, L_g}{N}_{\alpha_1,\ldots,\alpha_{g-1}} \equiv \sum_{n_1,\ldots,n_{g-1}}^{N} \binom{L_g}{N - \sum_{i=1}^{g-1} n_i} \prod_{i=1}^{g-1} \left(\frac{1+\alpha_i(-1)^{n_i}}{2}\right) \binom{L_i}{n_i}$$

Intuition:

# of possible ways to distribute N particles among g subsystems of size L_i with fixed parity α_i

4 GS Properties

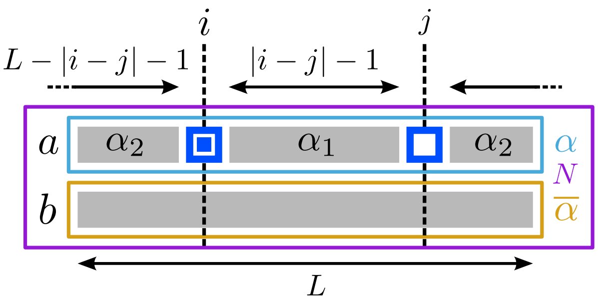
More interesting:

Greens function

Intrachain GF:

$$\langle a_i^\dagger a_j \rangle = \binom{L, L}{N}_\alpha^{-1} \left[A_{+1,-\alpha} - A_{-1,\alpha} \right]$$

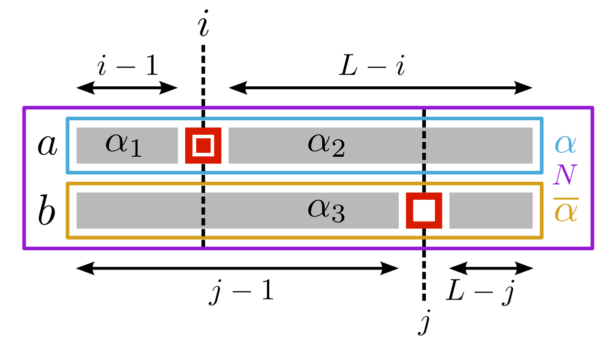
$$\text{where} \quad A = \binom{j-i-1, L-j+i-1, L}{N-1}$$



Interchain GF:

$$\langle a_i^\dagger b_j \rangle = \binom{L, L}{N}_\alpha^{-1} \left[B_{\alpha,+1,+1} + B_{-\alpha,-1,-1} - B_{\alpha,+1,-1} - B_{-\alpha,-1,+1} \right]$$

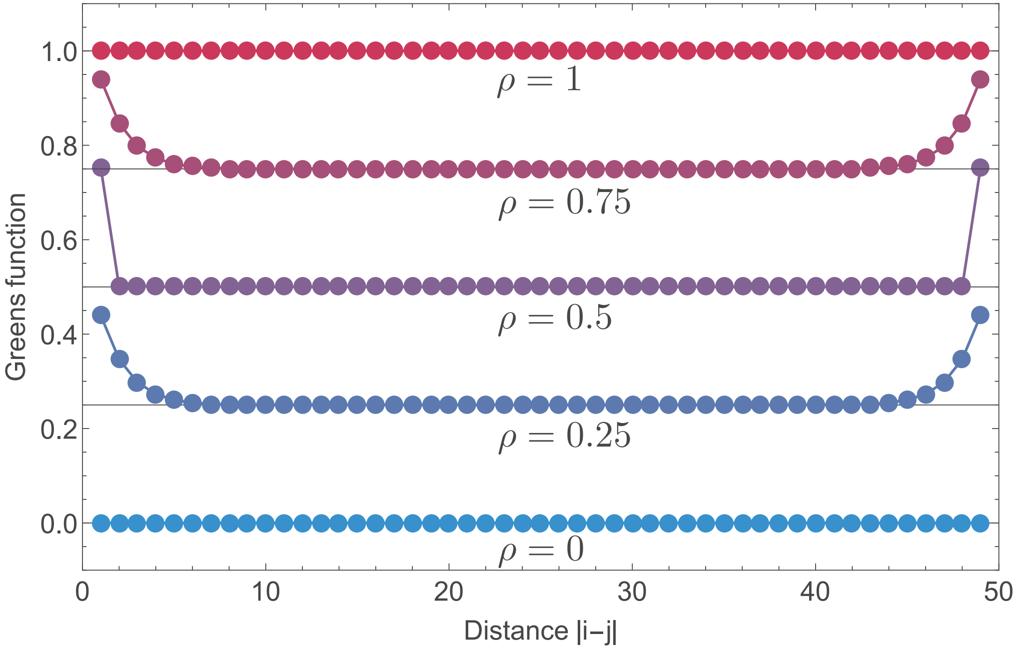
$$\text{where} \quad B = \binom{i-1, L-i, j-1, L-j}{N-1}$$

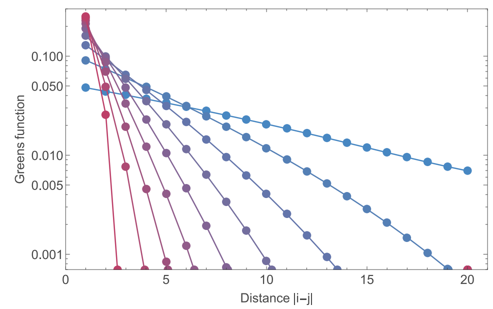


4 GS Properties

Greens Function:

Vanishes exponentially in the bulk & revival at the edges ...





We conclude:

Edge modes in the ground states!

5 Time-reversal Symmetry

Consider perturbations within a fixed particle-number sector:

When is the two-fold GS degeneracy lifted?

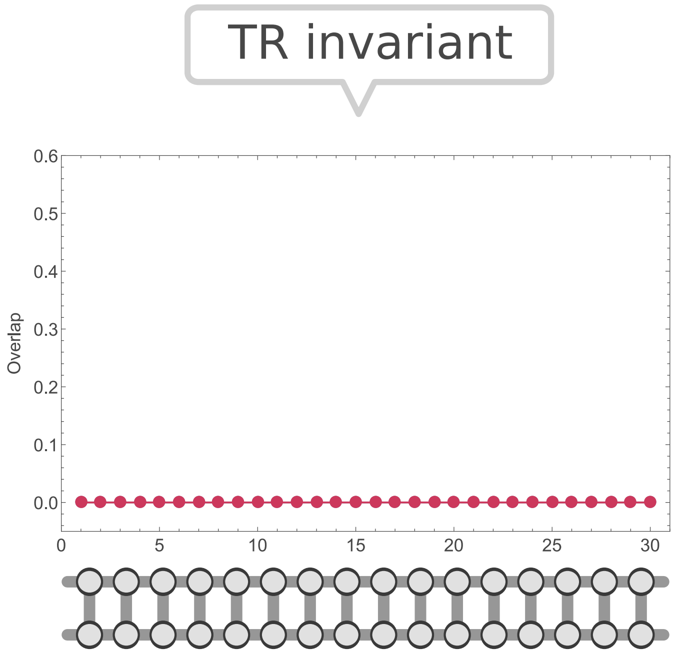
⇒ Subchain-parity violating interchain hopping:

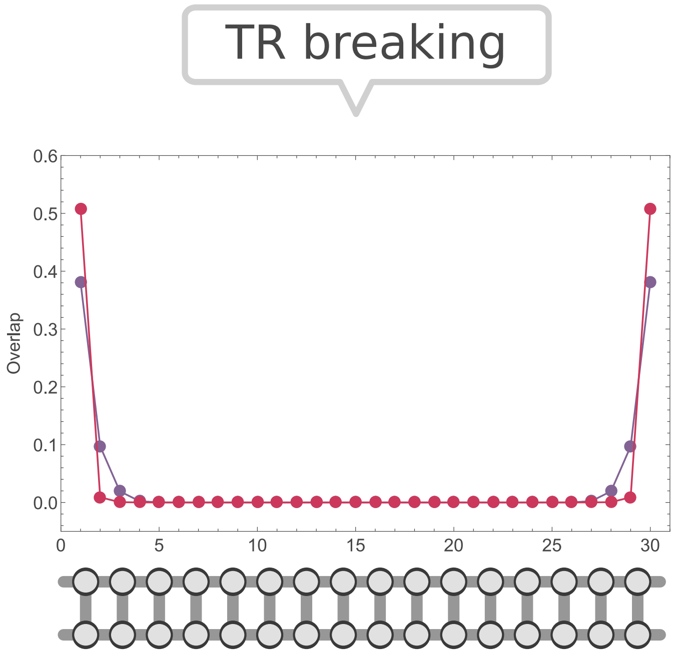
$$\langle -\alpha |\, a_i^\dagger b_i + b_i^\dagger a_i \,| \alpha \rangle \qquad \langle -\alpha |\, i a_i^\dagger b_i - i b_i^\dagger a_i \,| \alpha \rangle$$

TR invariant

TR breaking

5 Time-reversal Symmetry





We conclude:

No mixing for small, time-reversal invariant perturbations!

6 Excitations

Derivation of Excitations?

Jordan-Wigner Transformation:

Pauli matrices $\sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Raising operators $\sigma^+ = 1/2\left(\sigma^x + i\sigma^y\right)$

Lowering operators $\sigma^- = 1/2\left(\sigma^x - i\sigma^y\right)$

$$a_j, b_j \quad \mapsto \rho_{2L}\left(a_j\right) = \left[\prod_{k=1}^{j-1} \sigma_k^z\right] \sigma_j^+, \quad \rho_{2L}\left(b_j\right) = \left[\prod_{k=1}^{L+j-1} \sigma_k^z\right] \sigma_j^+$$

$$a_j^\dagger, b_j^\dagger \quad \mapsto \rho_{2L}\left(a_j^\dagger\right) = \left[\prod_{k=1}^{j-1} \sigma_k^z\right] \sigma_j^-, \quad \rho_{2L}\left(b_j^\dagger\right) = \left[\prod_{k=1}^{L+j-1} \sigma_k^z\right] \sigma_j^-$$

Result:

⇒ Equivalent Spin-Ladder (PBC):

$$H = \sum_{i=1}^{L} \left(H_i^a + H_i^b + H_i^{ab}\right) \qquad \in \mathcal{B}\left(\bigotimes_{i=1}^{2L} \mathbb{C}_i^2\right)$$

6 Single-Particle Excitations

Ansatz for small particle numbers:

Bethe Ansatz

Example:

Non-interacting single particle sector:

(↪ isotropic Heisenberg chain)

$$|\Psi\rangle = \sum_n a(n) |n\rangle$$

$a_n^\dagger |0\rangle$

→ Can be lifted to any filling sector

⇒ Quadratic low-energy spectrum:

$$E(k) = 2\left[1 - \frac{e^{ik} + e^{-ik}}{2}\right] = 4\sin^2\frac{k}{2}$$

⇒ gapless

⇒ Free Magnons as low-energy excitations:

$$|k; N = 1, \alpha = -1\rangle = \frac{1}{\sqrt{L}} \sum_{n=1}^{L} e^{ikn} |n\rangle \quad \text{with} \quad k \in \frac{2\pi}{L}\{0, 1, \ldots, L-1\}$$

⑥ Why do we care?

Note:

Quantum memory ↔ Ground state space

Quantum computation ↔ Braiding zero-modes

Compare:

Kitaev's Chain → gapped

Our model → gapless ... but:

Adiabatic Theorem

⇒ Gap closes algebraically (single-particle excitations):

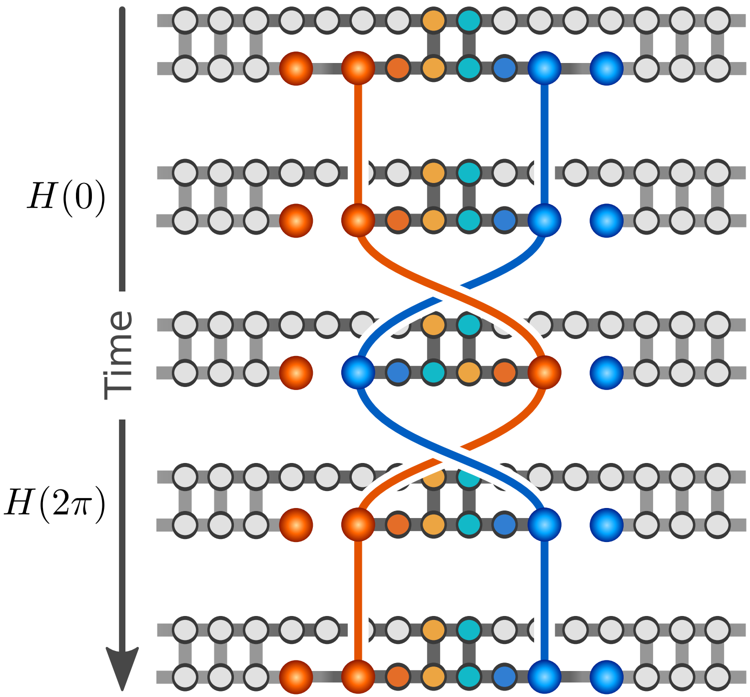
$$\Delta E \propto \frac{1}{L^2} \quad \text{for} \quad L \to \infty$$

⇒ Allows for generalised definition of braiding ...

7 Braiding & Statistics

$H(\varphi)$

Idea:
Braid edge-modes on subchain by adiabatic deformation of Hamiltonian



$\Longrightarrow$

Result:
Non-abelian Holonomy

$$|00\rangle_\ominus \xrightarrow{0\to\varphi\to2\pi} |11\rangle_\ominus$$
$$|01\rangle_\ominus \xrightarrow{0\to\varphi\to2\pi} |10\rangle_\ominus$$

$\Downarrow$

Non-abelian statistics:
Majorana Modes = Ising Anyons

The End

Thank you for your attention.