Majorana Modes in an interacting & number-conserving Theory

in preparation

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CO.CO.MAT
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Motivation

Kitaev's Majorana Chain

1 Motivation

$$\{a_i, a_j\} = 0 = \left\{a_i^\dagger, a_j^\dagger\right\} \quad \text{and} \quad \left\{a_i, a_j^\dagger\right\} = \delta_{i,j}$$

Paradigmatic Model:

Open chain of spinless fermions ↪ Kitaev Chain

$$H = -\sum_{i=1}^{L-1} \left[w a_i^\dagger a_{i+1} - |\Delta| a_i a_{i+1} + \text{h.c.}\right] - \mu \sum_{i=1}^{L} \left(a_i^\dagger a_i - \frac{1}{2}\right)$$

Majorana Fermions:

"Real & Imaginary part of fermionic generators"

$$c_{2i-1} \equiv a_i + a_i^\dagger \quad \text{and} \quad c_{2i} \equiv i\left(a_i^\dagger - a_i\right) \quad \text{for} \quad i = 1, \ldots, L$$

⇒ Self-adjoint fermions:

$$\{c_l, c_m\} = 2\delta_{l,m} \qquad \text{and} \qquad c_l = c_l^\dagger$$

Kitaev, A. Unpaired Majorana fermions in quantum wires. Physics-Uspekhi 131 (2007)

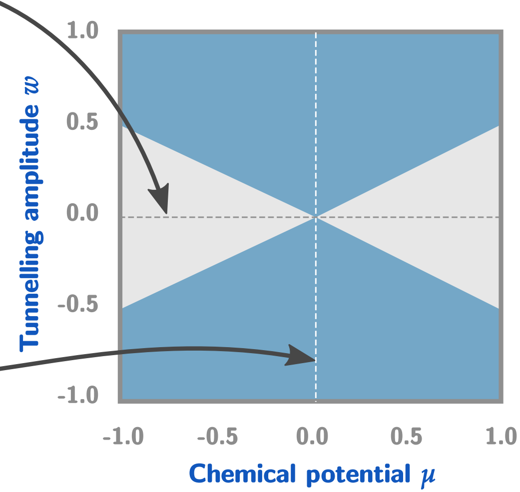
1 Motivation

$$a_i = \frac{1}{2}\left(c_{2i-1} + i c_{2i}\right) \quad \text{and} \quad a_i^\dagger = \frac{1}{2}\left(c_{2i-1} - i c_{2i}\right)$$

Trivial phase:

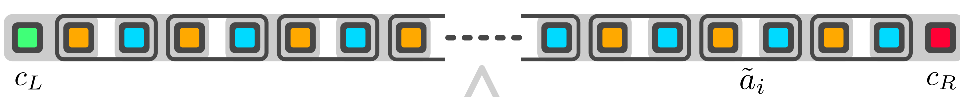
Dominant chemical potential

$$H = -\mu \frac{i}{2} \sum_{i=1}^{L} c_{2i-1} c_{2i}$$



Topological phase:

Dominant hopping & pairing



$$H = i w \sum_{i=1}^{L-1} c_{2i} c_{2i+1}$$

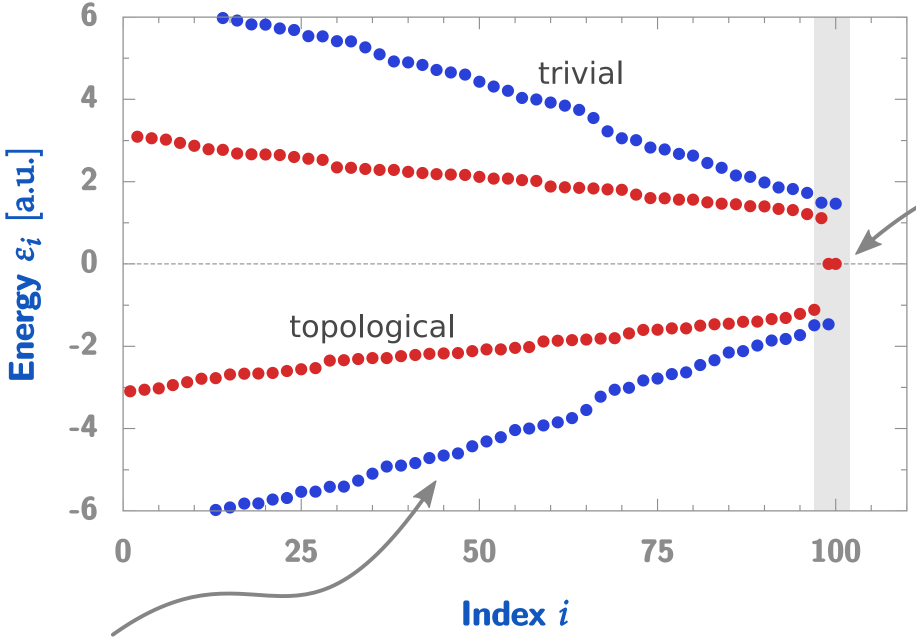
$$\tilde{a}_i \equiv \frac{1}{2}\left(c_{2i} + i c_{2i+1}\right) \quad \text{and} \quad \tilde{a}_i^\dagger \equiv \frac{1}{2}\left(c_{2i} - i c_{2i+1}\right)$$

⇒ Zero-energy edge mode

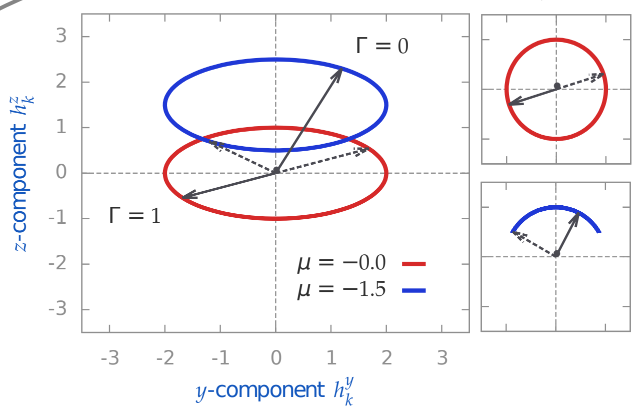
1 Motivation

Spectrum of Kitaev Chain:

Stable zero-energy modes in the topological phase:



Zero-energy modes = Winding Number



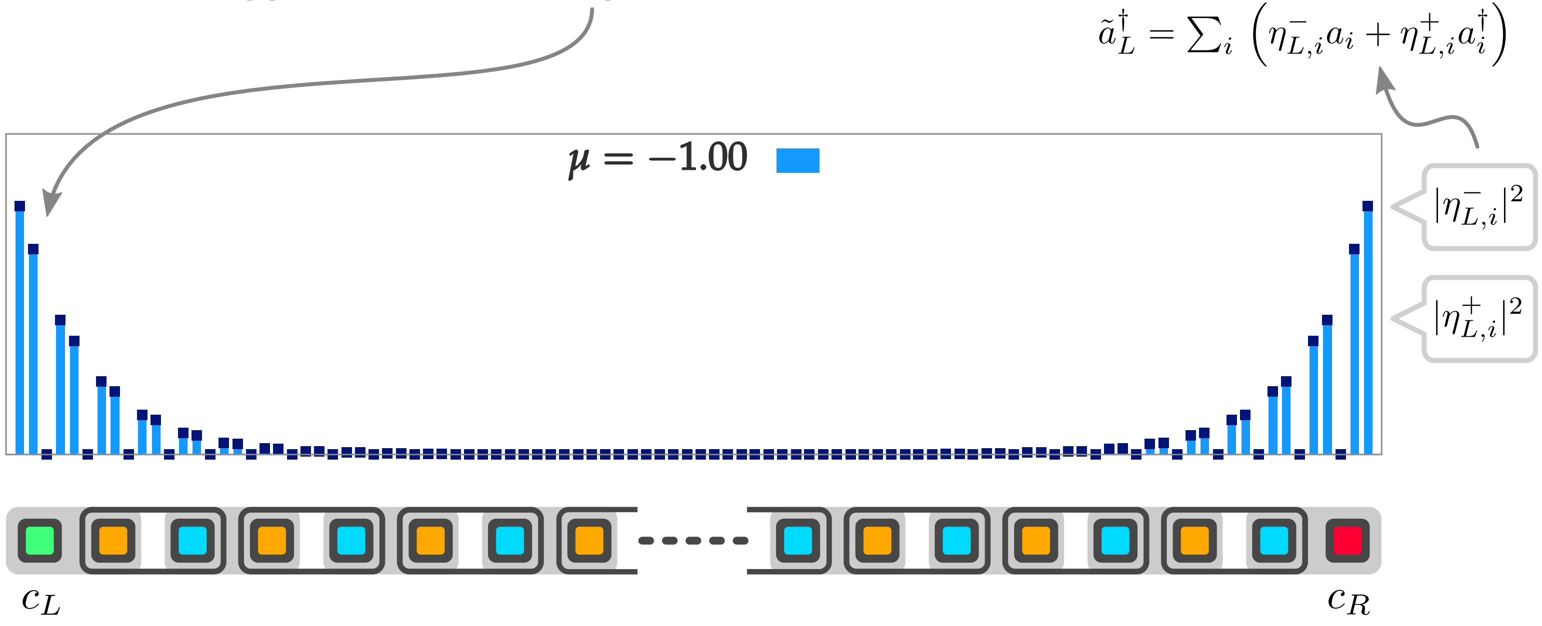
Static disorder → perturbed spectrum

1 Motivation

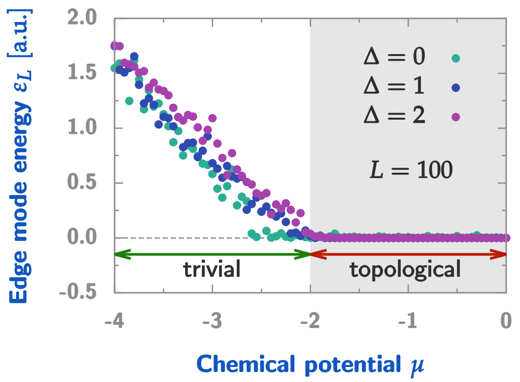
Wave functions:

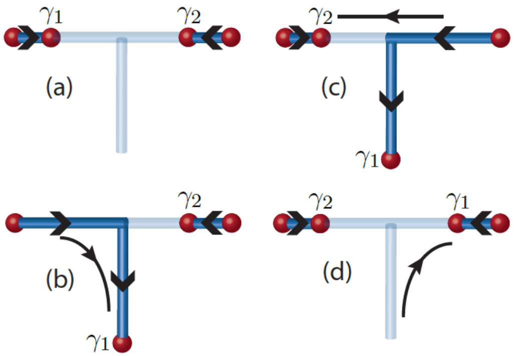
Zero-energy modes = Edge modes

$$\tilde{a}_L^\dagger = \sum_i \left(\eta^-_{L,i} a_i + \eta^+_{L,i} a_i^\dagger\right)$$



1 Why should we care?





Topological Phase:
Topological invariant protects ground state degeneracy

&

Anyonic Statistics:
Localized edge modes obey non-abelian anyonic statistics

↓

Inherently robust way of
storing & manipulating quantum information

⇒ Topological quantum memory & computer

J. Alicea et al. (2011). Non-Abelian statistics and topological quantum information processing in 1D wire networks. Nature Physics, 7(5), 412–417.

Beyond Mean Field?

$$H = -\sum_{i=1}^{L-1}\left[w a_i^\dagger a_{i+1} - |\Delta| a_i a_{i+1} + \text{h.c.}\right] - \mu \sum_{i=1}^{L}\left(a_i^\dagger a_i - \frac{1}{2}\right)$$

Beyond Mean Field?

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… is not particle-number conserving.

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A particle-number conserving Theory?

Beyond Mean Field?

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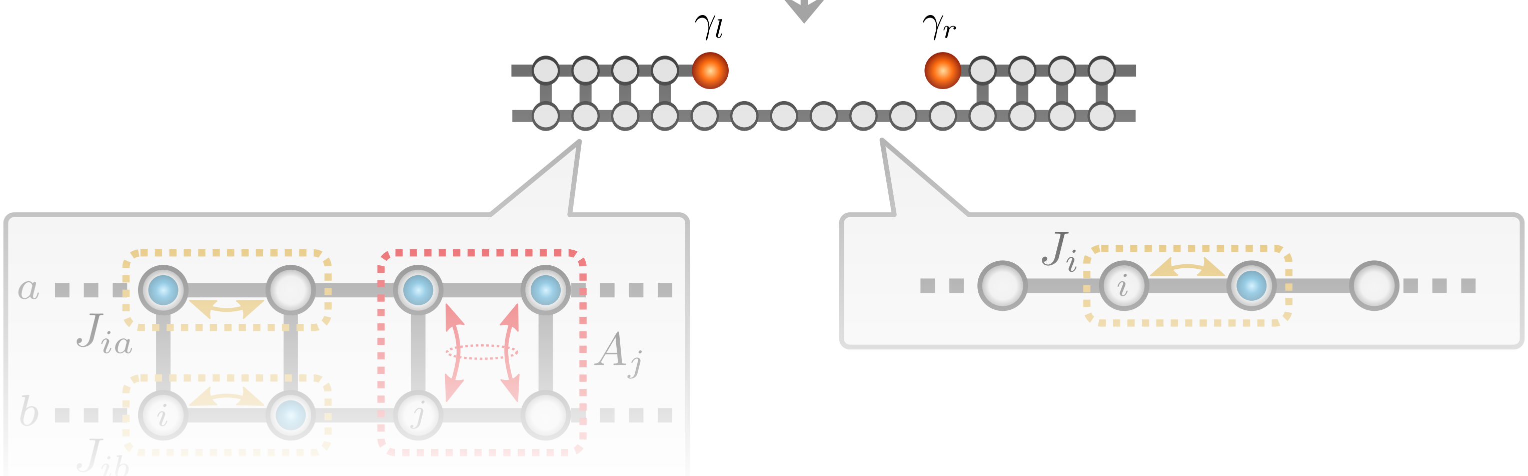
↓

A particle-number conserving Theory?

↓



2 Beyond Mean Field?

Previous work in this context:

→ M. Cheng and H.-H. Tu (2011). Physical Review B, 84(9), 094503.
Majorana edge states in interacting two-chain ladders of fermions.

→ J. D. Sau et al. (2011). Physical Review B, 84(14), 144509.
Number conserving theory for topologically protected degeneracy in one-dimensional fermions.

→ L. Fidkowski et al. (2011). Physical Review B, 84(19), 195436.
Majorana zero modes in one-dimensional quantum wires without long-ranged superconducting order.

→ J. Ruhman et al. (2014). arXiv:1412.3444
Topological States in a One-Dimensional Fermi Gas with Attractive Interactions.

Bosonization

→ C. V. Kraus et al. (2013). Physical Review Letters, 111(17), 173004.
Majorana Edge States in Atomic Wires Coupled by Pair Hopping.

Numerical (DMRG)

→ G. Ortiz et al. (2014). arXiv:1407.3793
Many-body characterization of topological superconductivity: The Richardson-Gaudin-Kitaev chain.

Long-Range

⇒ Here: Short-range interacting Theory
Exact ground state
Majorana Modes on edges

② Beyond Mean Field?

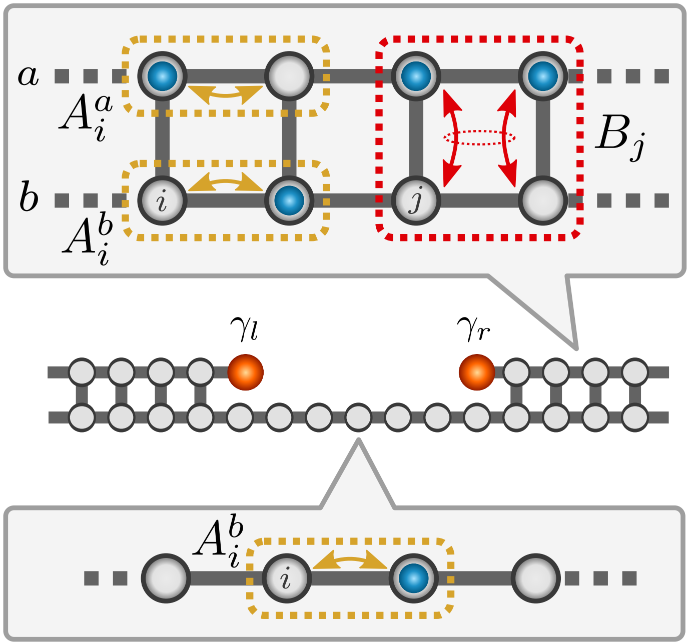
Our Model:

→ Double Chain

→ Spinless Fermions on Sites

→ Intra- & Interchain Interactions

$$H = H^a + H^b + H^{ab} \equiv \sum_{i=1}^{L-1} H_i$$

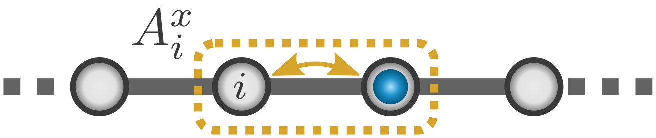


Symmetries:

→ Particle-number

→ Time-reversal / Subchain-Parity

2 Beyond Mean Field?



Intrachain Interactions:

Single-particle hopping & NN density-density interactions

$$A_i^x = x_i^\dagger x_{i+1} + x_{i+1}^\dagger x_i \qquad \text{for } x = a, b$$

$$H^x = \sum_{i=1}^{L-1} A_i^x \left(\mathbb{1} + A_i^x\right)$$

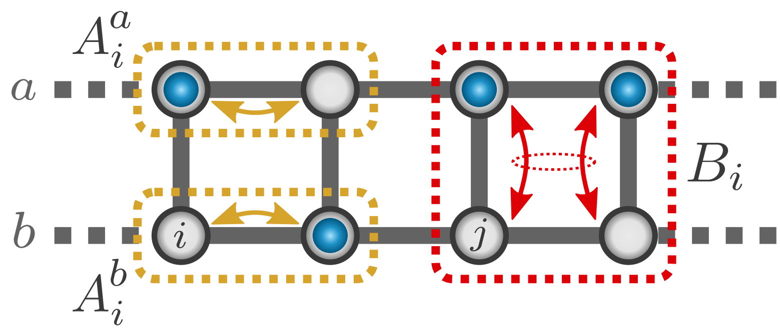
$$J_{x,i} = \frac{1}{\sqrt{2}}\left(x_i^\dagger + x_{i+1}^\dagger\right)\left(x_i - x_{i+1}\right)$$

$$H^x = \sum_{i=1}^{L-1} J_{x,i}^\dagger J_{x,i}$$

Expanded form:

$$H_i^x = x_i x_{i+1}^\dagger + x_{i+1} x_i^\dagger + n_i^x\left(\mathbb{1} - n_{i+1}^x\right) + n_{i+1}^x\left(\mathbb{1} - n_i^x\right)$$

② Beyond Mean Field?



Interchain Interaction:

Pair-hopping & Pair-density-density interactions

$$B_i = a_i^\dagger a_{i+1}^\dagger b_i b_{i+1} + b_i^\dagger b_{i+1}^\dagger a_i a_{i+1}$$

$$H^{ab} = \sum_{i=1}^{L-1} B_i \left(\mathbb{1} + B_i\right)$$

Expanded form:

$$H_i^{ab} = a_i^\dagger a_{i+1}^\dagger b_i b_{i+1} + b_i^\dagger b_{i+1}^\dagger a_i a_{i+1} + n_i^a n_{i+1}^a (\mathbb{1} - n_i^b)(\mathbb{1} - n_{i+1}^b) + n_i^b n_{i+1}^b (\mathbb{1} - n_i^a)(\mathbb{1} - n_{i+1}^a)$$

③ Exact Ground State

$$H^x = \sum_{i=1}^{L-1} A_i^x \left(\mathbb{1} + A_i^x\right) \qquad H^{ab} = \sum_{i=1}^{L-1} B_i \left(\mathbb{1} + B_i\right)$$

Facts:

→ Each term in the Hamiltonian is a positive operator

→ Sum of positive operators = positive operator

Derivation of Exact Ground State (GS):

→ If zero-energy ⇒ GS = simultaneous kernel of local terms

→ If kernel trivial ⇒ Finite-energy GS (not accessible)

3 Exact Ground State

Do the math ...

GS = Equal-weight superposition with fixed
total particle number & subchain parity

$$|N;\alpha\rangle = \mathcal{N} \sum_{M,\,(-1)^M=\alpha} |M\rangle_a \, |N-M\rangle_b$$

with

$$|M\rangle_x = \sum_{\boldsymbol{m}\in\{0,1\}^L,\,|\boldsymbol{m}|=M} |\boldsymbol{m}\rangle_x$$

Degeneracy for open boundary conditions:

Two GS for each particle number sector:

N even	even	odd
	even	odd
N odd	even	odd
	odd	even

Note:

Does not work for periodic boundary conditions in all sectors!

4 Symmetry Protection

Consider perturbations within a fixed particle-number sector:

When is the two-fold GS degeneracy lifted?

⇒ Subchain-parity violating interchain hopping:

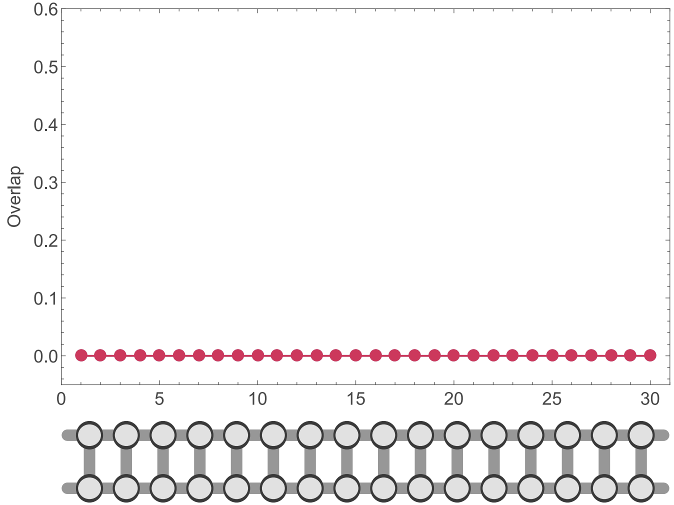
$$\langle -\alpha |\, a_i^\dagger b_i + b_i^\dagger a_i \,| \alpha \rangle \qquad \langle -\alpha |\, i a_i^\dagger b_i - i b_i^\dagger a_i \,| \alpha \rangle$$

TR invariant

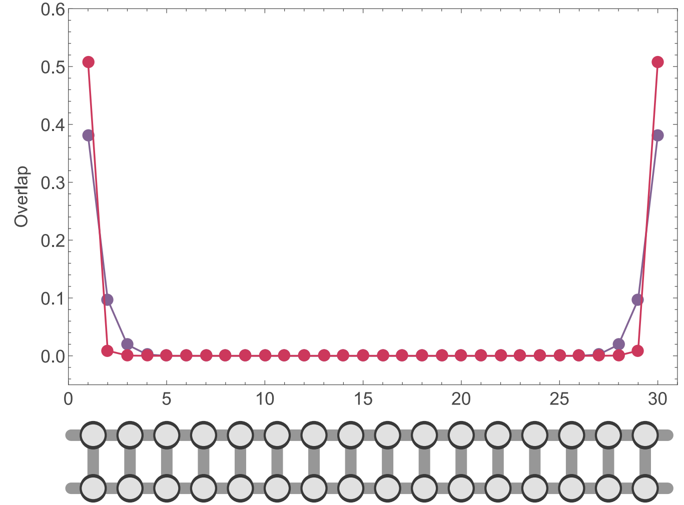
TR breaking

4 Symmetry Protection

TR invariant



TR breaking



We conclude:

No mixing for small, time-reversal invariant perturbations!

5 GS Properties

Evaluation of arbitrary correlators:

Pure Combinatorics!

Generalization:

Parity-split Binomial Coefficients (PsBC)

$$\binom{L_1,\dots,L_g}{\alpha_1,\dots,\alpha_{g-1}}_N \equiv \sum_{n_1,\dots,n_{g-1}}^{N} \binom{L_g}{N-\sum_{i=1}^{g-1} n_i} \prod_{i=1}^{g-1} \left[\frac{1+\alpha_i(-1)^{n_i}}{2}\right] \binom{L_i}{n_i}$$

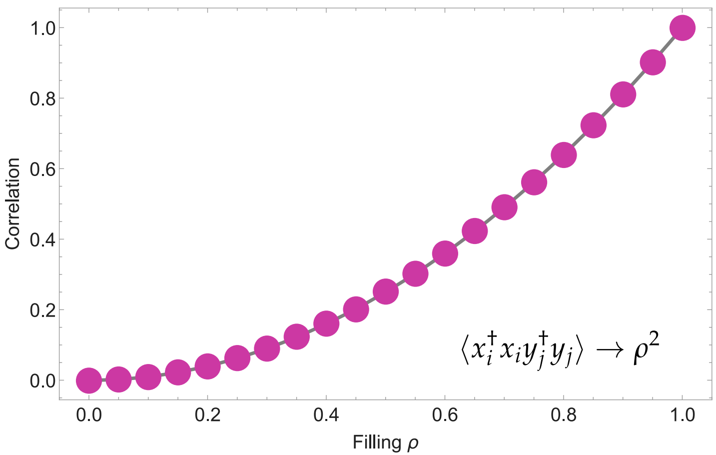
Intuition:

# of possible ways to distribute N particles among g subsystems of size L_i with fixed parity α_i

5 GS Properties

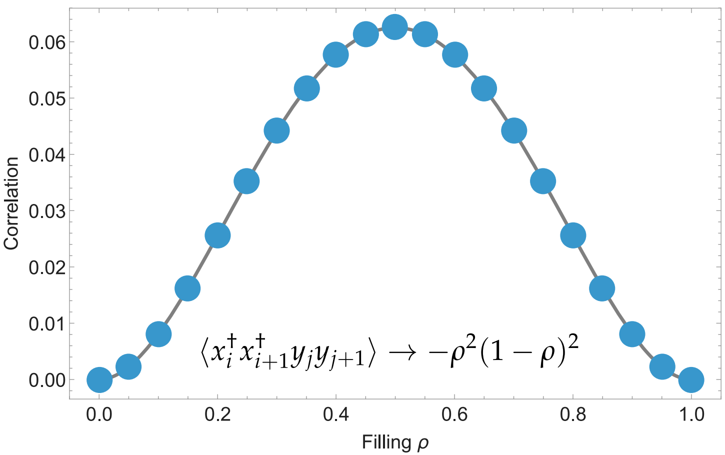
Simple Example 1:
Density-density correlation

$$\langle x_i^\dagger x_i y_j^\dagger y_j \rangle = \begin{cases} \binom{L-1,L-1}{-\alpha}_{N-2} / \binom{L,L}{\alpha}_N & \text{for} \quad x \neq y \\ \binom{L-2,L}{\alpha}_{N-2} / \binom{L,L}{\alpha}_N & \text{for} \quad x = y \end{cases}$$



Simple Example 2:
Pair correlation

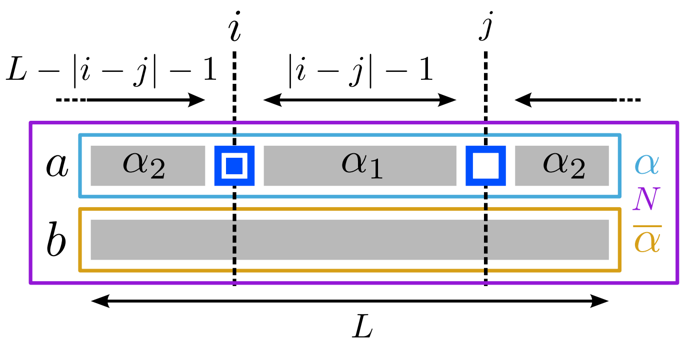
$$\langle x_i^\dagger x_{i+1}^\dagger y_j y_{j+1} \rangle = \begin{cases} \binom{L-2,L-2}{\alpha}_{N-2} / \binom{L,L}{\alpha}_N & \text{for} \quad x \neq y \\ \binom{L-4,L}{\alpha}_{N-2} / \binom{L,L}{\alpha}_N & \text{for} \quad x = y \end{cases}$$



5 GS Properties

More interesting:

Greens function $\langle a_i^\dagger a_j \rangle$



Intrachain GF:

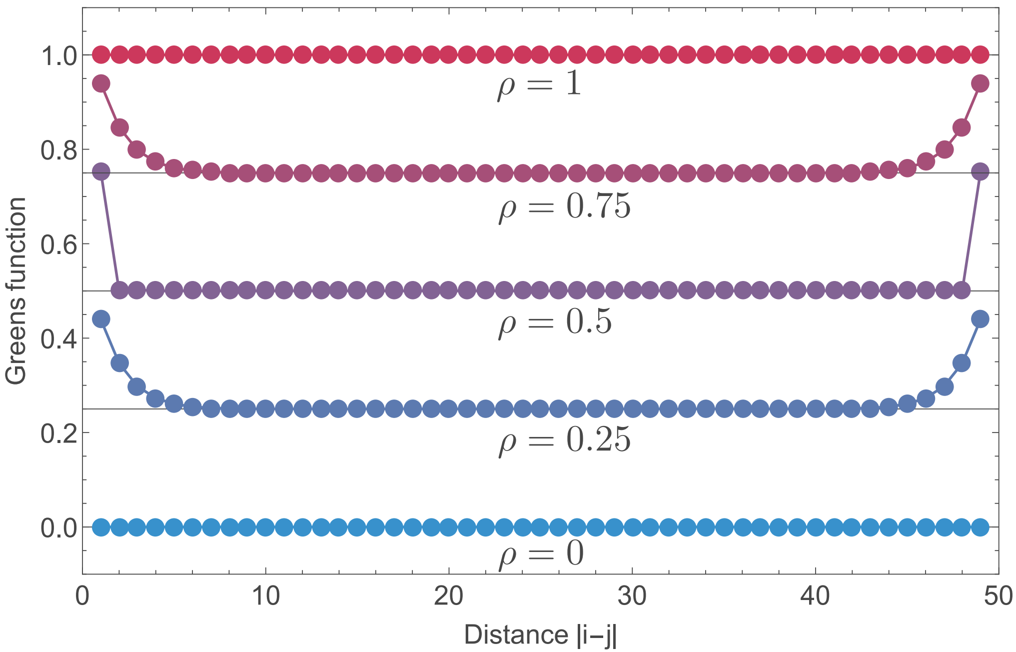
$$\langle a_i^\dagger a_j \rangle = \mathcal{N}_{L,N,\alpha}^{-1} \left[(+1,-\alpha)_1 - (-1,\alpha)_1 \right]$$

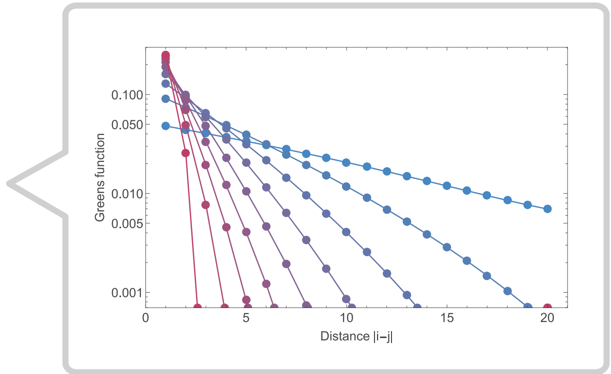
where $\mathcal{N}_{L,N,\alpha} \equiv \binom{L,L}{N}_\alpha$ and $(\bullet)_1 \equiv \binom{j-i-1, L-j+i-1, L}{\bullet}_{N-1}$

5 GS Properties

Greens Function:

Vanishes exponentially in the bulk & revival at the edges ...

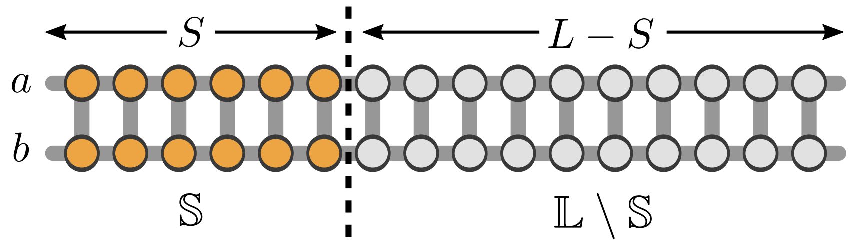




We conclude:

Edge modes in the ground states!

6 Entanglement Spectrum



Definition:

Entanglement Spectrum (ES) $\{\xi_{n,\beta}\}$ defined via Schmidt-decomposition:

$$|N;\alpha\rangle = \mathcal{N} \sum_{M,\,(-1)^M=\alpha} |M\rangle_a\,|N-M\rangle_b \equiv \sum_{n=0}^{N}\sum_{\beta=\pm1} e^{-\frac{\xi_{n,\beta}}{2}}\,|N-n;\alpha\beta\rangle_{\mathbb{L}\setminus\mathbb{S}} \otimes |n;\beta\rangle_{\mathbb{S}}$$

ONB in subsystems

Conventional wisdom:

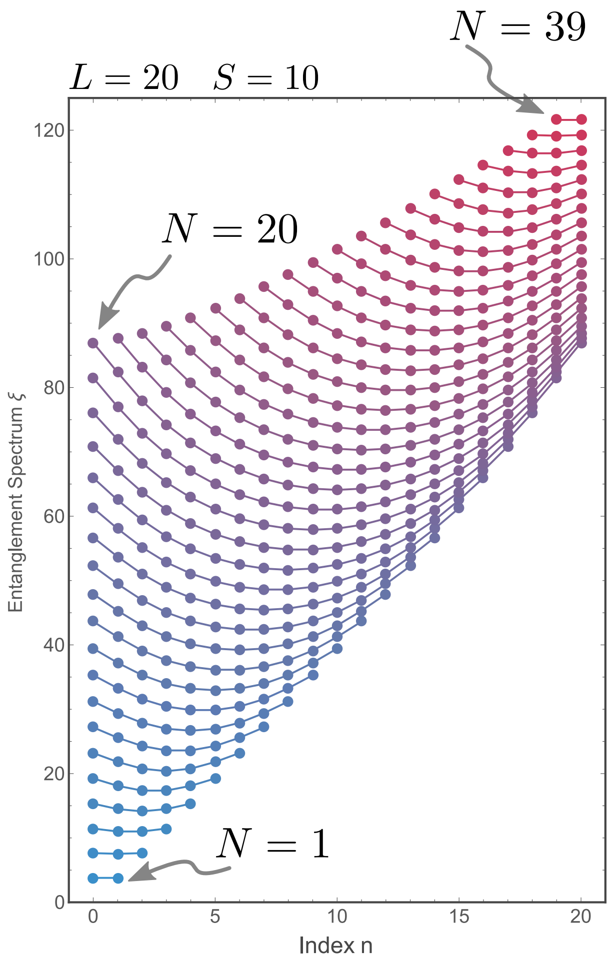
Topologically protected GS ⇔ Degenerate ES (cf. Majorana chain)

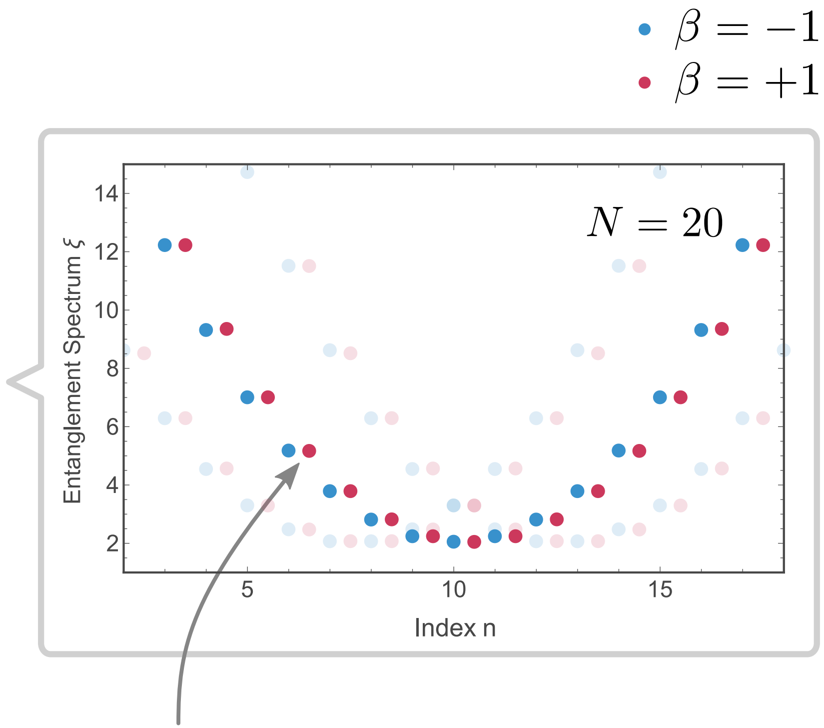
Exact result:

The ES $\{\xi_{n,\beta}\}$ is indexed by (n,β) with $\beta \in \{+1,-1\}$, $\max\{0, N+2S-2L\} \leq n \leq \min\{N, 2S\}$

$$\xi_{n,\beta} = -\ln\left[\binom{L-S,\,L-S}{\alpha\beta}_{N-n}\binom{S,S}{\beta}_{n}\Big/\binom{L,L}{\alpha}_{N}\right]$$

⑥ Entanglement Spectrum





⇒ 2-fold degenerate

(cf. Kitaev chain ✓)

6 Entanglement Entropy

Definition:

Entanglement Entropy (EE) defined via partial trace $\rho_{\mathbb{S}} \equiv \mathrm{Tr}_{\mathbb{L}\setminus\mathbb{S}}[\rho]$

$$S^{\mathrm{ent}}[\mathbb{S}] \equiv S^{\mathrm{vN}}[\rho_{\mathbb{S}}] = -\,\mathrm{Tr}\left[\rho_{\mathbb{S}} \ln \rho_{\mathbb{S}}\right]$$

Intuition:

EE quantifies Entanglement between subsystems

("entanglement measure")

Computation:

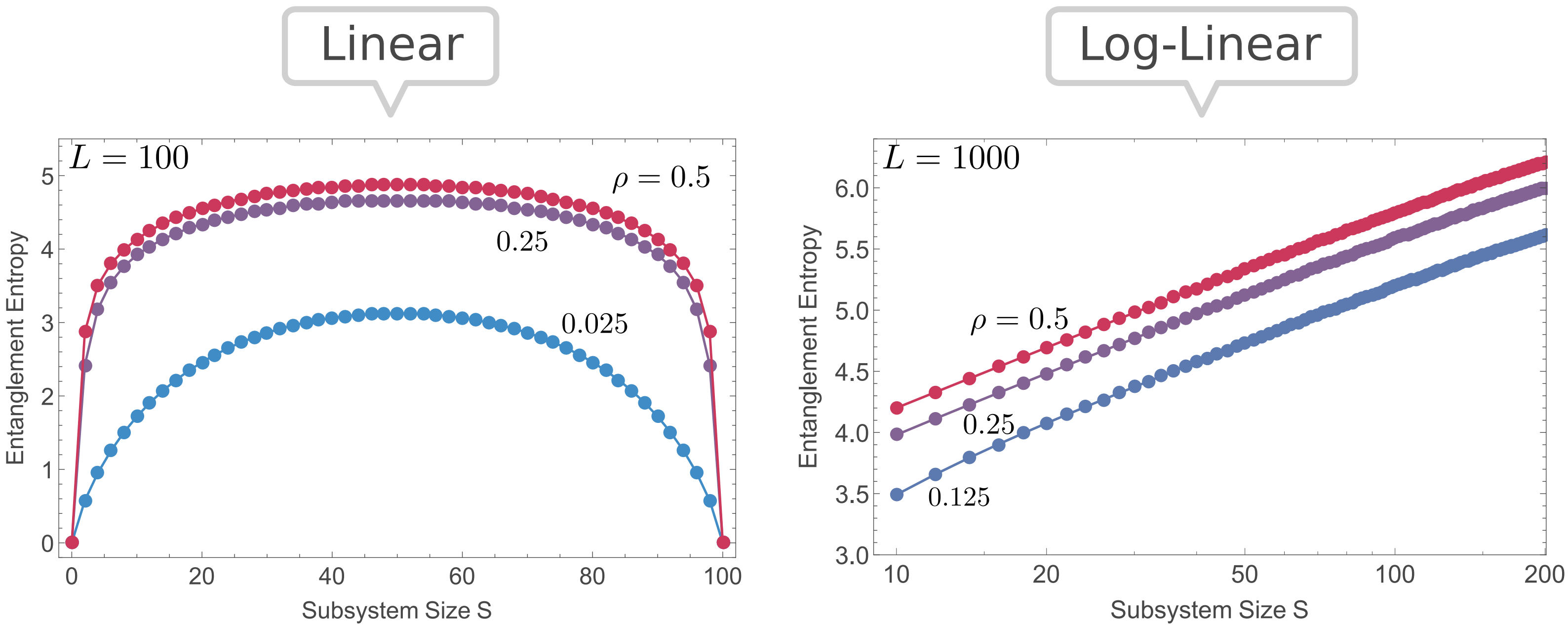
EE is a functional of the ES:

$$S^{\mathrm{ent}}[\mathbb{S}] = \sum_i e^{-\xi_i} \xi_i$$

⑥ Entanglement Entropy

Dependence of Subsystem Size:

EE grows logarithmically with Subsystem Size S:



Conclusion:

Area-Law with logarithmic Corrections ⇒ Critical 1D System!

7 Excitations

Derivation of Excitations?

Jordan-Wigner Transformation:

Pauli matrices $\sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$$a_j, b_j \quad \mapsto \rho_{2L}\left(a_j\right) = \left[\prod_{k=1}^{j-1} \sigma_k^z\right] \sigma_j^+, \quad \rho_{2L}\left(b_j\right) = \left[\prod_{k=1}^{L+j-1} \sigma_k^z\right] \sigma_j^+$$

$$a_j^\dagger, b_j^\dagger \quad \mapsto \rho_{2L}\left(a_j^\dagger\right) = \left[\prod_{k=1}^{j-1} \sigma_k^z\right] \sigma_j^-, \quad \rho_{2L}\left(b_j^\dagger\right) = \left[\prod_{k=1}^{L+j-1} \sigma_k^z\right] \sigma_j^-$$

Raising operators

$\sigma^+ = 1/2\left(\sigma^x + i\sigma^y\right)$

Lowering operators

$\sigma^- = 1/2\left(\sigma^x - i\sigma^y\right)$

Result:

⇒ Equivalent Spin-Ladder (PBC):

$$H = \sum_{i=1}^{L} \left(H_i^a + H_i^b + H_i^{ab}\right) \qquad \in \mathcal{B}\left(\bigotimes_{i=1}^{2L} \mathbb{C}_i^2\right)$$

7 Single-Particle Excitations

Ansatz for small particle numbers:
Bethe Ansatz

Example:
Non-interacting single particle sector:
(↪ isotropic Heisenberg chain)

$$|\Psi\rangle = \sum_n a(n)\,|n\rangle \quad \leftarrow a_n^\dagger |0\rangle$$

→ Can be lifted to any filling sector

⇒ Quadratic low-energy spectrum:

$$E(k) = 2\left[1 - \frac{e^{ik} + e^{-ik}}{2}\right] = 4\sin^2\frac{k}{2}$$

⇒ gapless

⇒ Free Magnons as low-energy excitations:

$$|k; N=1, \alpha=-1\rangle = \frac{1}{\sqrt{L}}\sum_{n=1}^{L} e^{ikn}\,|n\rangle \quad \text{with} \quad k \in \frac{2\pi}{L}\{0, 1, \ldots, L-1\}$$

7 Why do we care?

Note:
Quantum memory ↔ Ground state space
Quantum computation ↔ Braiding zero-modes

Compare:
Kitaev's Chain → gapped ← Adiabatic Theorem
Our model → gapless ... but:

⇒ Gap closes algebraically (single-particle excitations):

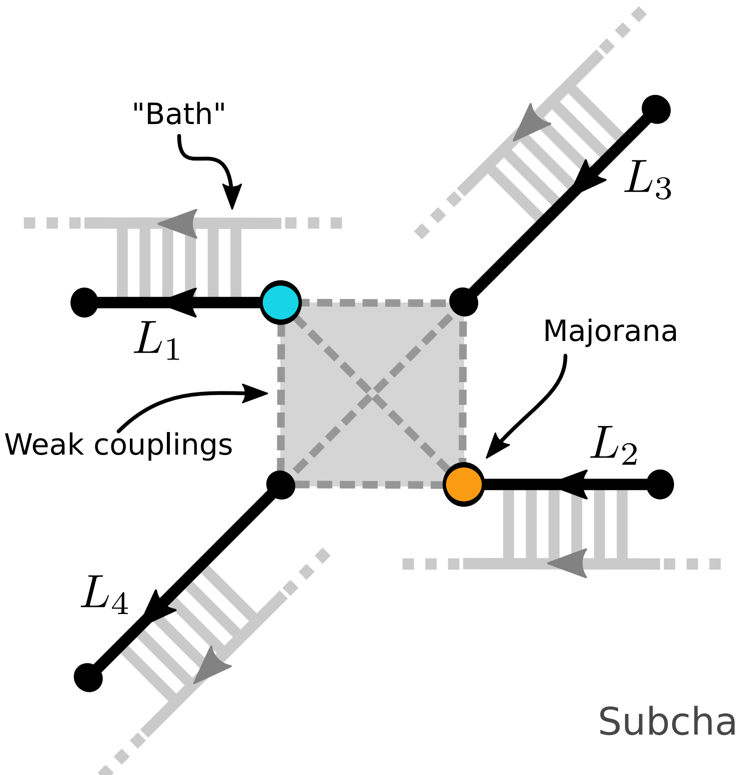
$$\Delta E \propto \frac{1}{L^2} \quad \text{for} \quad L \to \infty$$

⇒ Allows for generalised definition of braiding ...

8 Braiding & Statistics

Idea:

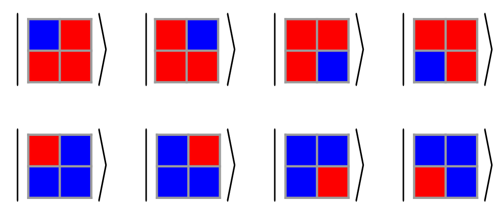
Braid edge-modes on subchains by adiabatic deformation of Hamiltonian



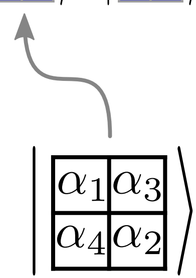
Low-energy sector:

Negative total parity

→ 8 ground states



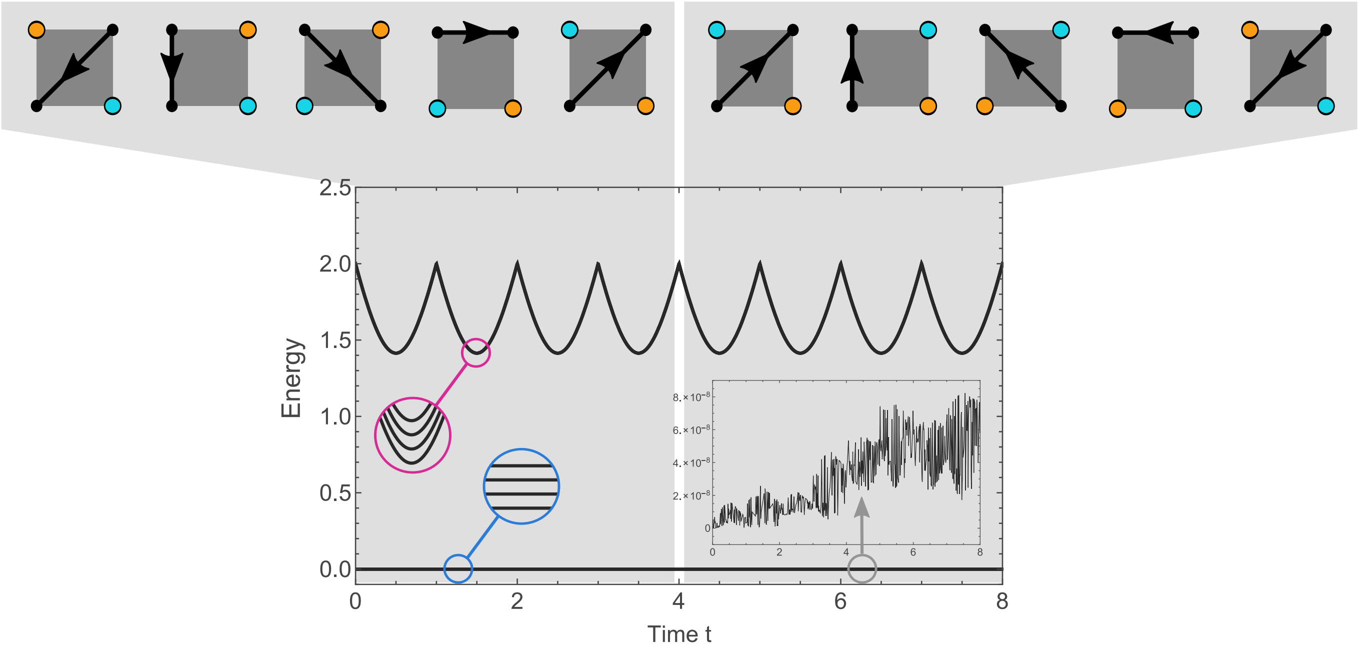
Subchain-parities:



8 Braiding & Statistics

Result:

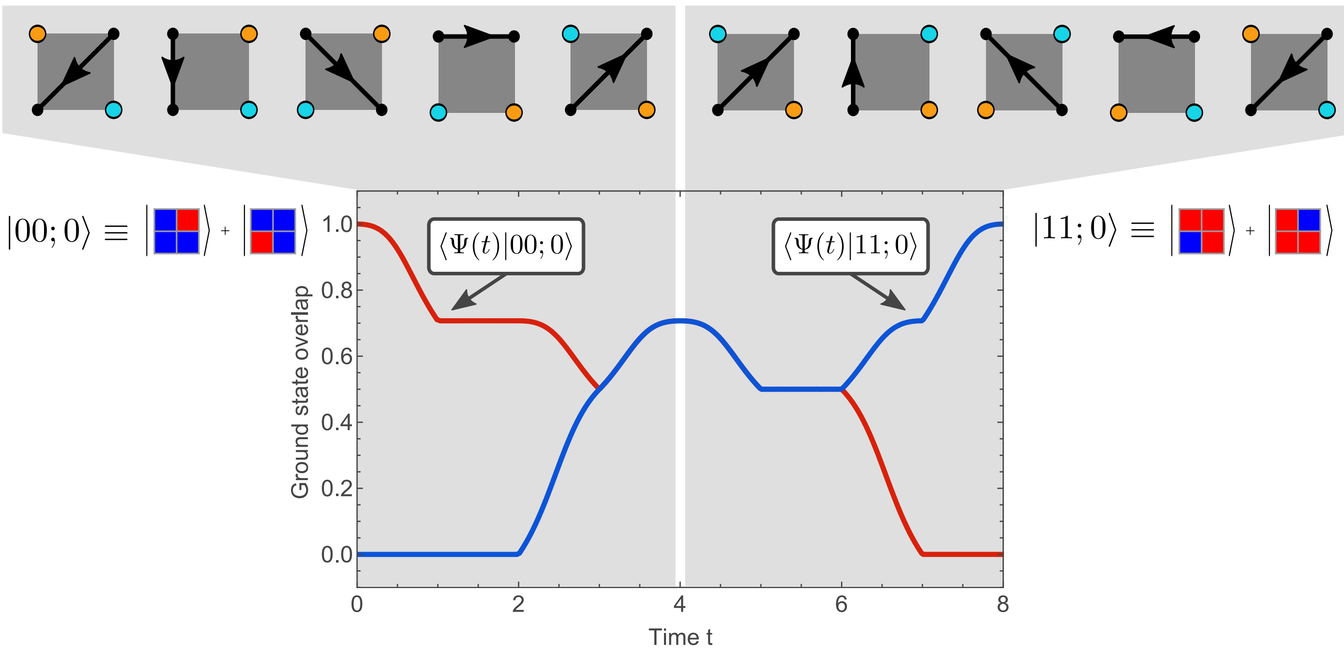
Non-abelian Holonomy:



8 Braiding & Statistics

Result:

Non-abelian Holonomy:



$\Longrightarrow$ Non-abelian statistics:

Edge Modes = Majorana Modes = Ising Anyons

The End

Thank you for your attention.