

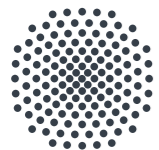
# Entanglement Transition in the Projective Transverse Field Ising Model

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Institute for Theoretical Physics III  
University of Stuttgart

Institute Seminar @ CNQO, University of Strathclyde (via Zoom)

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**University of Stuttgart**  
Germany



**European Research Council**  
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# PROLOGUE

# Random unitary circuits

PHYSICAL REVIEW B **98**, 205136 (2018)

## Quantum Zeno effect and the many-body entanglement transition

Yaodong Li,<sup>1</sup> Xiao Chen,<sup>2</sup> and Matthew P. A. Fisher<sup>1</sup>

<sup>1</sup>Department of Physics, University of California, Santa Barbara, California 93106, USA

<sup>2</sup>Kavli Institute for Theoretical Physics, University of California, Santa Barbara, California 93106, USA

PHYSICAL REVIEW B **100**, 134306 (2019)

Editors' Suggestion

## Measurement-driven entanglement transition in hybrid quantum circuits

Yaodong Li,<sup>1</sup> Xiao Chen,<sup>2</sup> and Matthew P. A. Fisher<sup>1</sup>

<sup>1</sup>Department of Physics, University of California, Santa Barbara, California 93106, USA

<sup>2</sup>Kavli Institute for Theoretical Physics, University of California, Santa Barbara, California 93106, USA

See also:

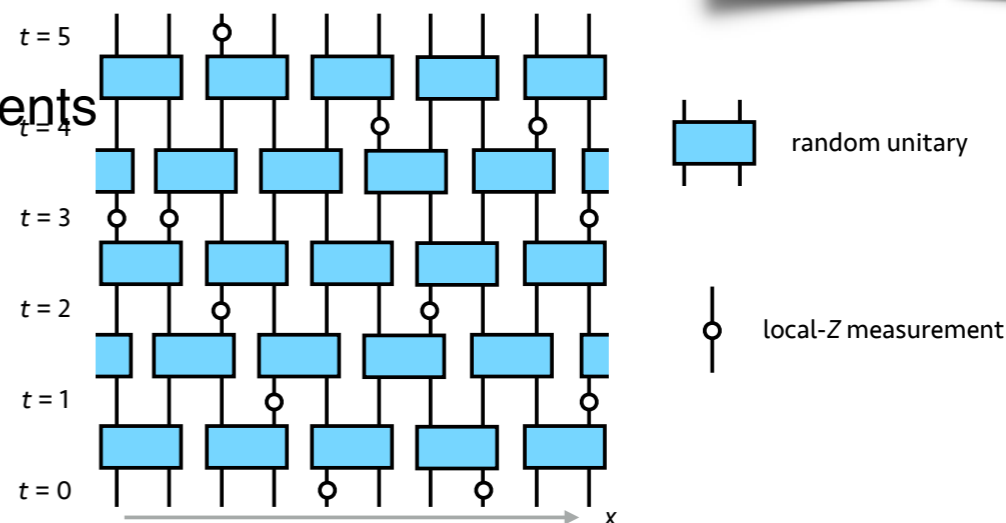
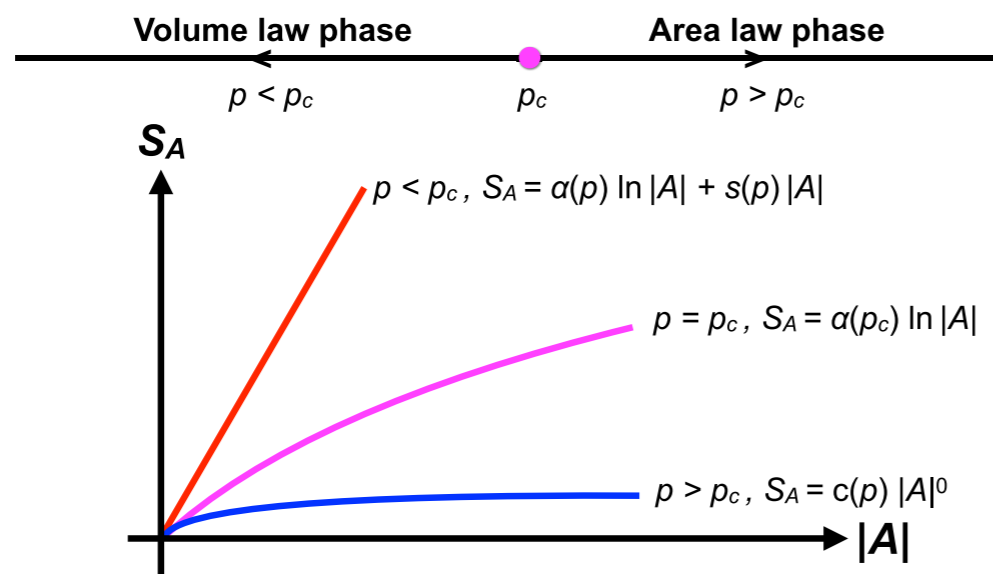
A. Chan et. al. (2018)

B. Skinner et. al. (2019)

M. Gullans et. al. (2019)

Y. Bao et. al. (2020)

- Random unitaries compete with projective measurements
- **Clifford circuits** described by **stabilizer formalism**  
-> Studied numerically for  $n > 500$  spins



- Unitaries: create entanglement  
-> **Volume-law**
- Measurements: destroy entanglement  
-> **Area-law**

**Question:** Can one discard the unitaries and still find entanglement transitions ?

# Random unitary projective circuits

PHYSICAL REVIEW B **102**, 094204 (2020)

Editors' Suggestion

## Entanglement transition in the projective transverse field Ising model

Nicolai Lang<sup>1</sup> and Hans Peter Büchler<sup>1</sup>

*Institute for Theoretical Physics III and Center for Integrated Quantum Science and Technology,  
University of Stuttgart, 70550 Stuttgart, Germany*

Answer: Yes!

See also:

## Entanglement phase transitions in measurement-only dynamics

Matteo Ippoliti,<sup>1</sup> Michael J. Gullans,<sup>2</sup> Sarang Gopalakrishnan,<sup>3,4</sup> David A. Huse,<sup>2,5</sup> and Vedika Khemani<sup>1</sup>

<sup>1</sup>*Department of Physics, Stanford University, Stanford, CA 94305, USA*

<sup>2</sup>*Department of Physics, Princeton University, Princeton, NJ 08544, USA*

<sup>3</sup>*Department of Engineering Science and Physics,  
CUNY College of Staten Island, Staten Island, NY 10314, USA*

<sup>4</sup>*Initiative for Theoretical Sciences, The Graduate Center, CUNY, New York, NY 10016, USA*

<sup>5</sup>*Institute for Advanced Study, Princeton, NJ 08540, USA*

PHYSICAL REVIEW RESEARCH **2**, 023288 (2020)

## Entanglement and dynamics of diffusion-annihilation processes with Majorana defects

Adam Nahum<sup>1</sup> and Brian Skinner<sup>2,3</sup>

<sup>1</sup>*Theoretical Physics, University of Oxford, Parks Road, Oxford OX1 3PU, England, United Kingdom*

<sup>2</sup>*Department of Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139, USA*

<sup>3</sup>*Department of Physics, Ohio State University, Columbus, Ohio 43210, USA*

## Measurement-induced topological entanglement transitions in symmetric random quantum circuits

Ali Lavasani, Yahya Alavirad, and Maissam Barkeshli

*Department of Physics, Condensed Matter Theory Center,  
University of Maryland, College Park, Maryland 20742, USA and  
Joint Quantum Institute, University of Maryland, College Park, Maryland 20742, USA*

## Measurement Protected Quantum Phases

Shengqi Sang<sup>1</sup> and Timothy H. Hsieh<sup>1</sup>

<sup>1</sup>*Perimeter Institute for Theoretical Physics, Waterloo, Ontario N2L 2Y5, Canada*

# The Projective Transverse field Ising Model (DTIM)

Paradigmatic model for quantum phase transitions:

## Transverse field Ising model

$$H = - \sum_i \sigma_i^z \sigma_{i+1}^z - h \sum_i \sigma_i^x$$

Non-commuting  
Hermitian generators

$\sigma_i^z \sigma_{i+1}^z$        $\sigma_i^x$

Non-commuting  
projective observables

**-> Projective transverse field Ising model**

# Simulation: The generic approach

## Gottesman-Knill-Theorem

Quantum circuits with ...

- Pauli measurements ✓
  - Pauli gates
  - Hadamard-, phase-, and CNOT-gates
- ... can be simulated efficiently on a classical computer

Clifford group

-> Stabilizer Formalism

D. Gottesman,  
*Stabilizer Codes and Quantum Error Correction*  
(1997)

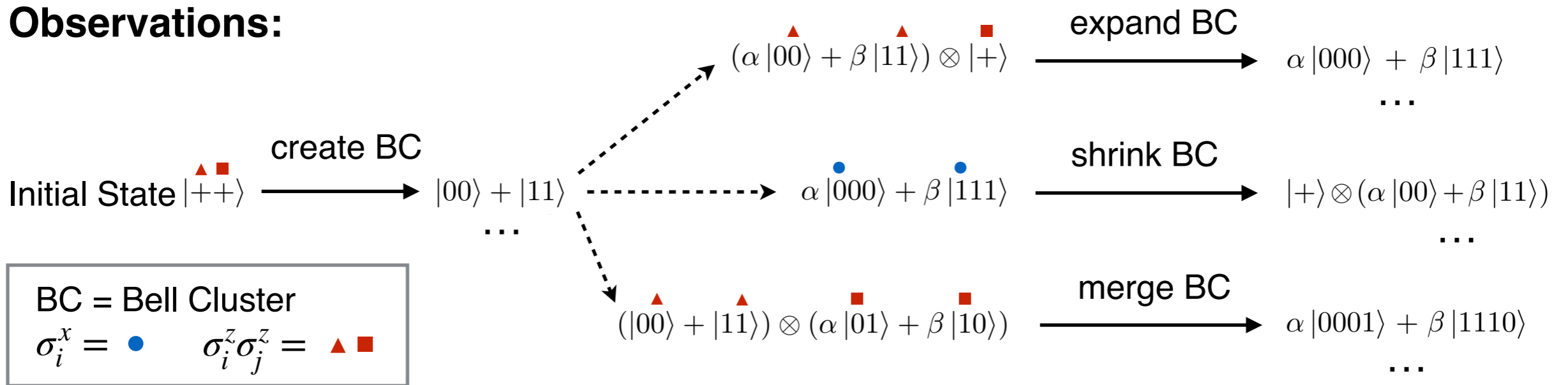
Nielsen and Chuang,  
*Quantum Computation and Quantum Information*

- Algebraic formalism to describe specific quantum states & gates
- Standard tool for random Clifford circuits
- Useful for up to ~1000 spins  
(fine in 1D, but only 30 x 30 on square lattice!)

More efficient approach that provides more insight ?

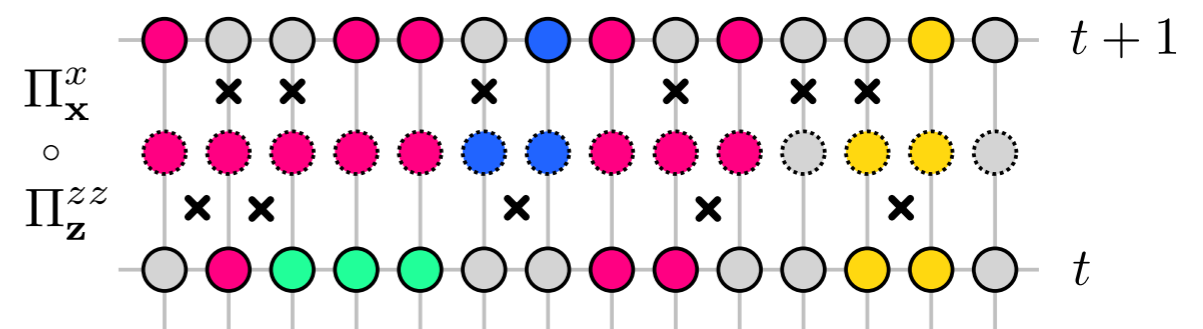
# Simulation: The specific approach

## Observations:



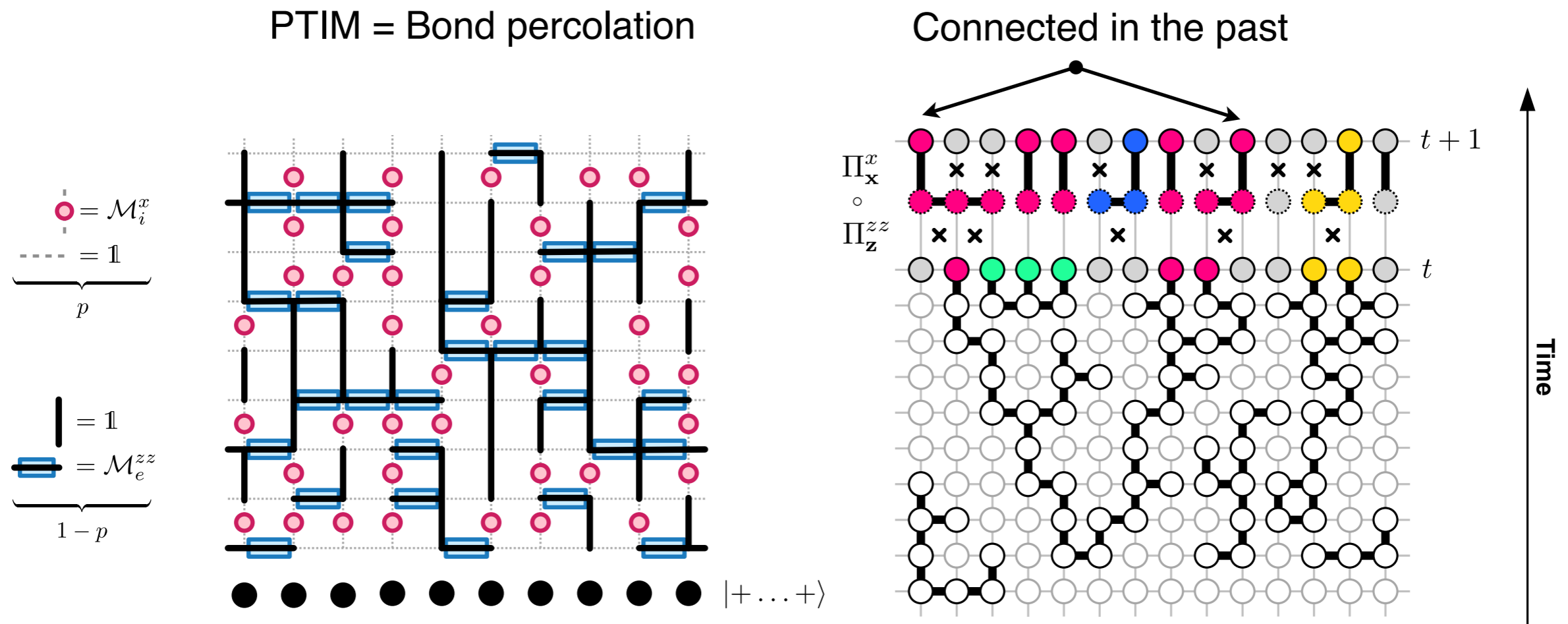
## -> Colored Cluster Model (CCM)

- Label independent spins with 0  $\circ$
- Label **Bell clusters** with integers  $n = 1, 2, 3, \dots$   $\bullet \bullet \bullet$
- $\sigma_i^x$ -measurement  
-> shrink Bell clusters (set spins to 0)
- $\sigma_i^z \sigma_j^z$ -measurements  
-> merge/create Bell clusters between spins



( Colors = Gauge degrees of freedom )

# Colored cluster model & Bond percolation



- CCM = **Non-local**, classical stochastic process
- **Bond percolation** on spacetime  $\rightarrow$  CCM on *boundary* of spacetime

**$\rightarrow$  Simulation more efficient than with stabilizers !**

# RESULTS

# Entanglement & Mutual information in 1D

**Entanglement entropy (EE)** of subsystem  $X$ :

$$S(X) = -\text{Tr} [\rho_X \log_2 \rho_X]$$

→ EE of subsystem with length  $l$ :

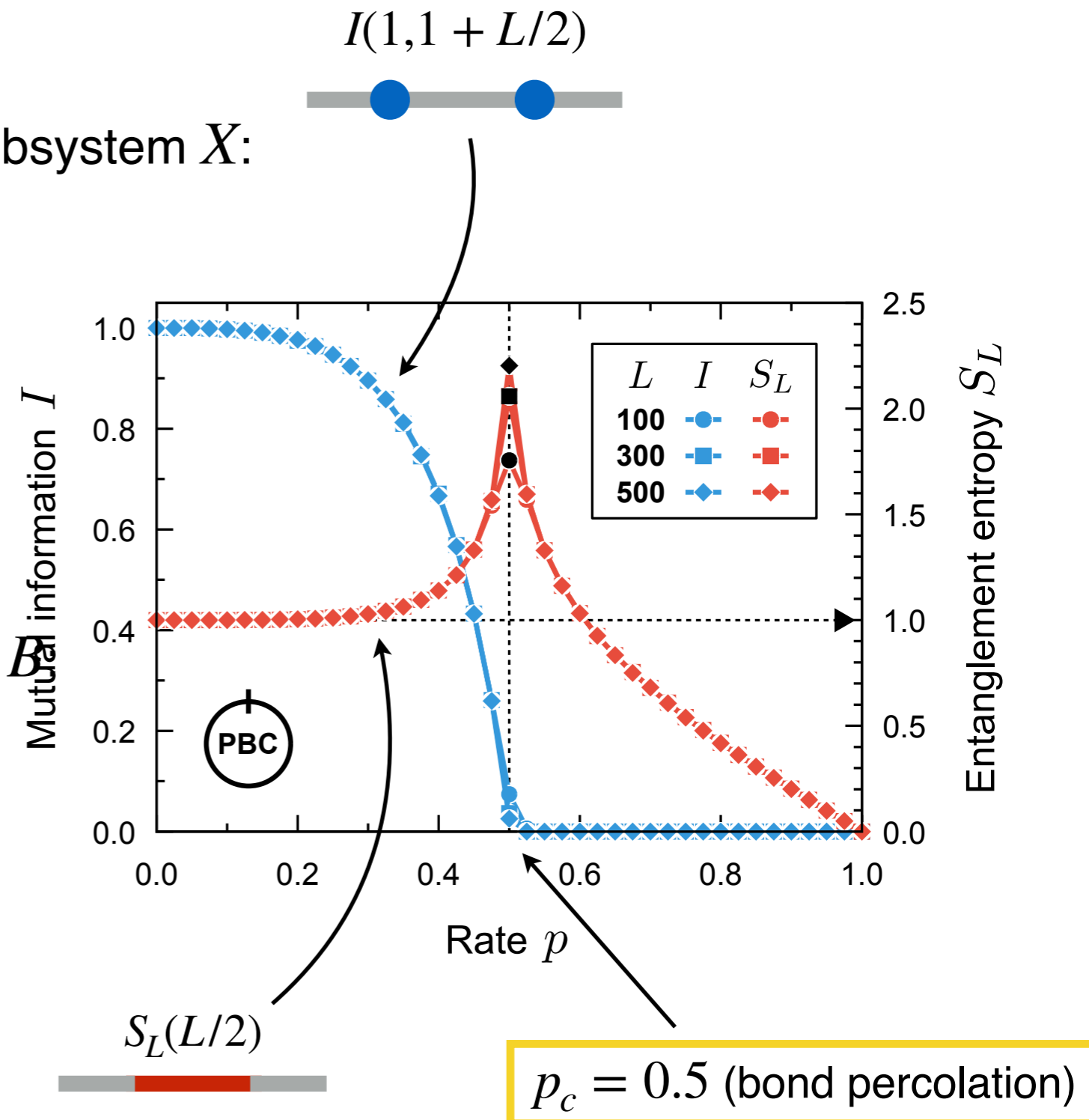
$$S_L(l)$$

**Mutual information (MI)** of  $A$  and  $B$

$$I(A, B) \equiv S(A) + S(B) - S(A \cup B)$$

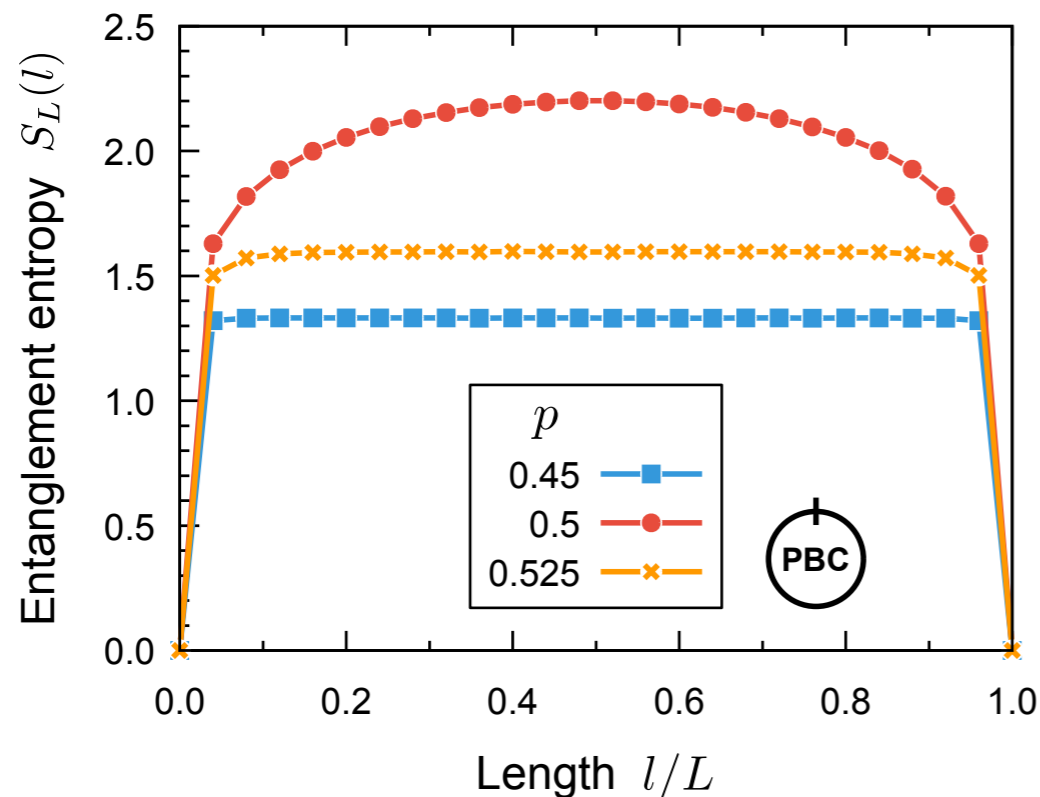
→ MI of two spins:

$$I(i, j) \equiv S(\{i\}) + S(\{j\}) - S(\{i, j\})$$



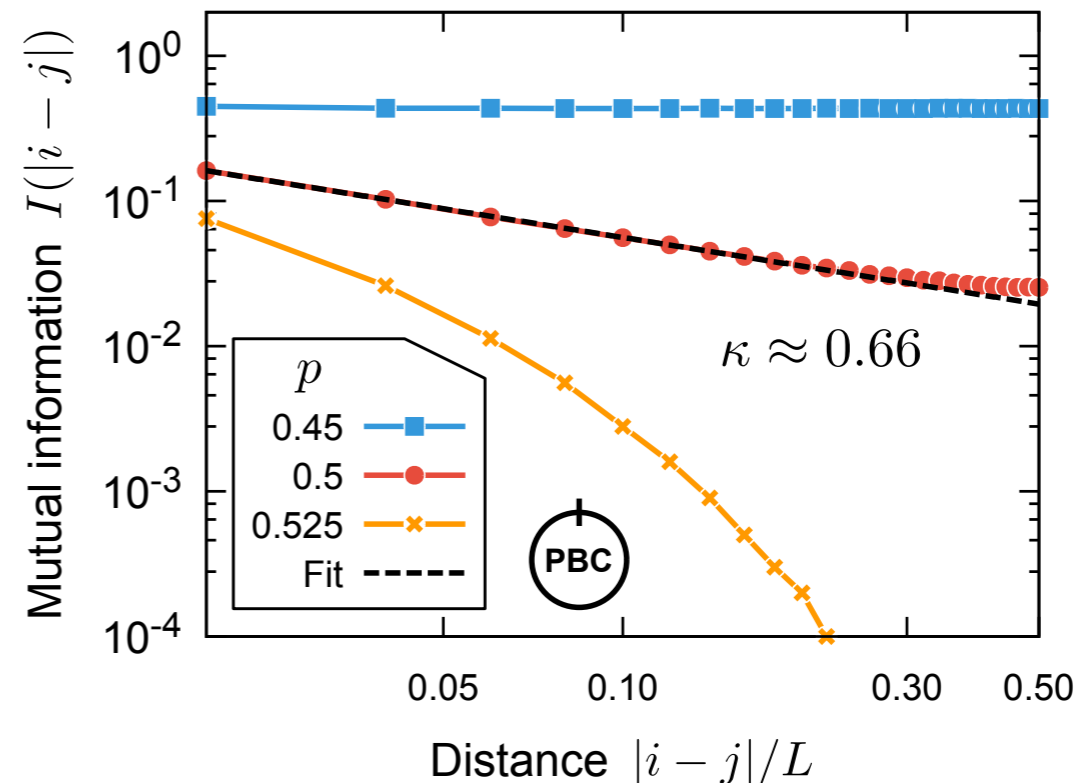
# Entanglement & Mutual information in 1D

Entanglement entropy  $S_L(l)$



- Area laws for both  $p < p_c$  and  $p > p_c$
- Logarithmic correction at criticality (see below)

Mutual information  $I(|i - j|)$



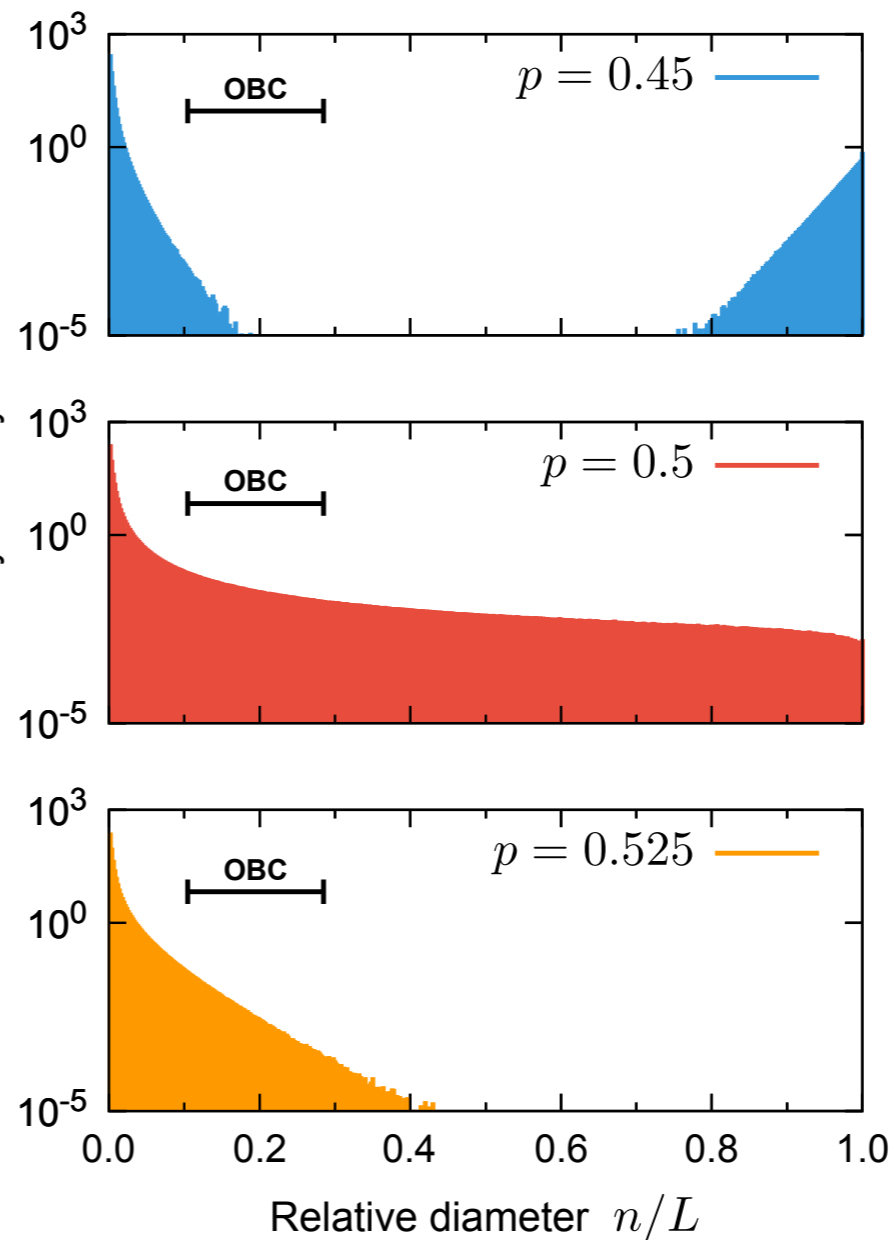
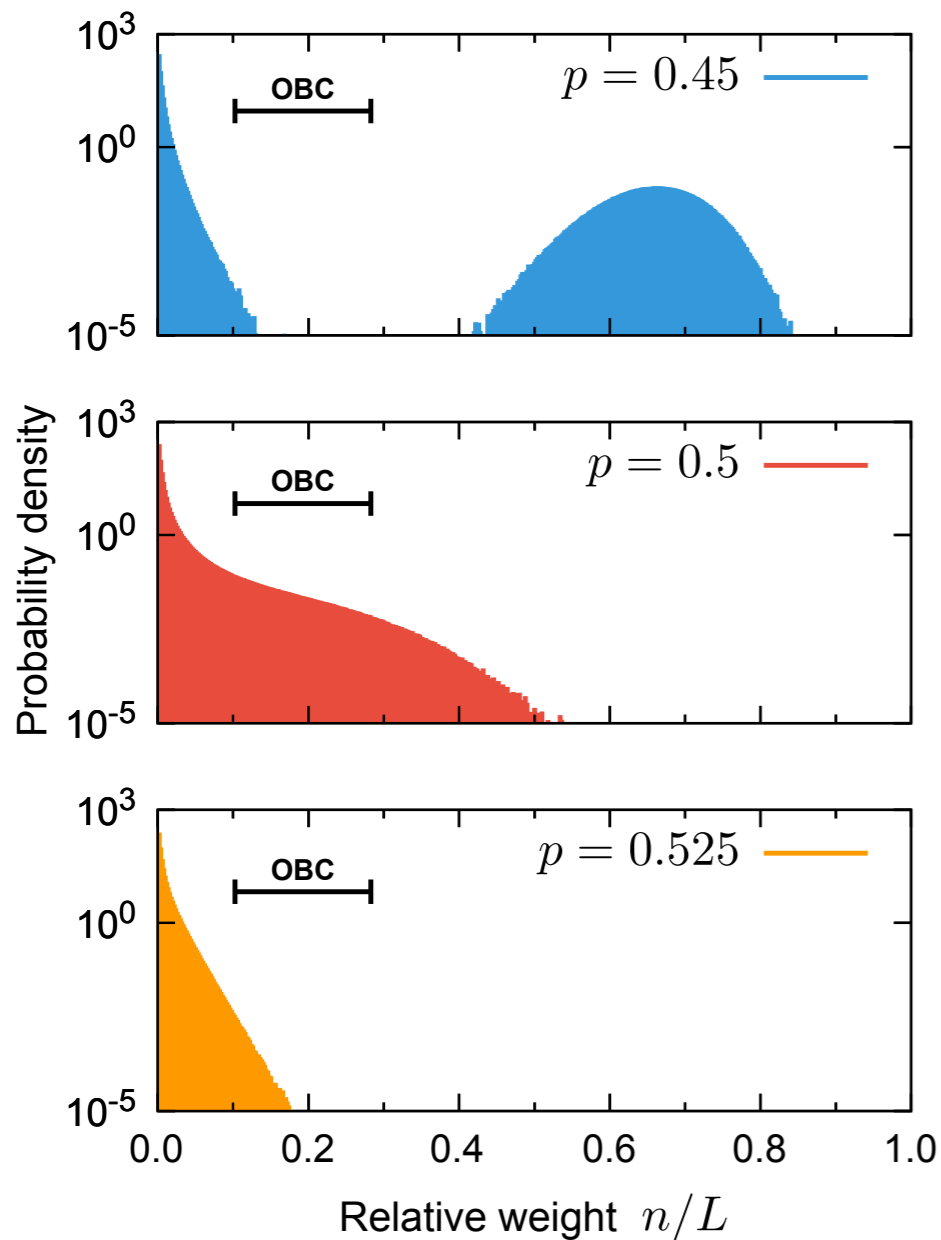
- Non-zero for  $p < p_c$
- Exponential decay for  $p > p_c$
- Power law at criticality:

$$I(|i - j|) \sim |i - j|^{-\kappa}, \quad \kappa \approx 0.66$$

# Cluster statistics in 1D

## Histogram of cluster weight

## Histogram of cluster diameter



$p < p_c$   
One dense global cluster

$p = p_c$   
Sparse clusters on  
all length scales

$p > p_c$   
Small, local clusters

# Critical point: CFT description?

1D critical systems are often described by **Conformal Field Theories (CFT)**

- Scaling of entanglement entropy is universal<sup>†</sup>:

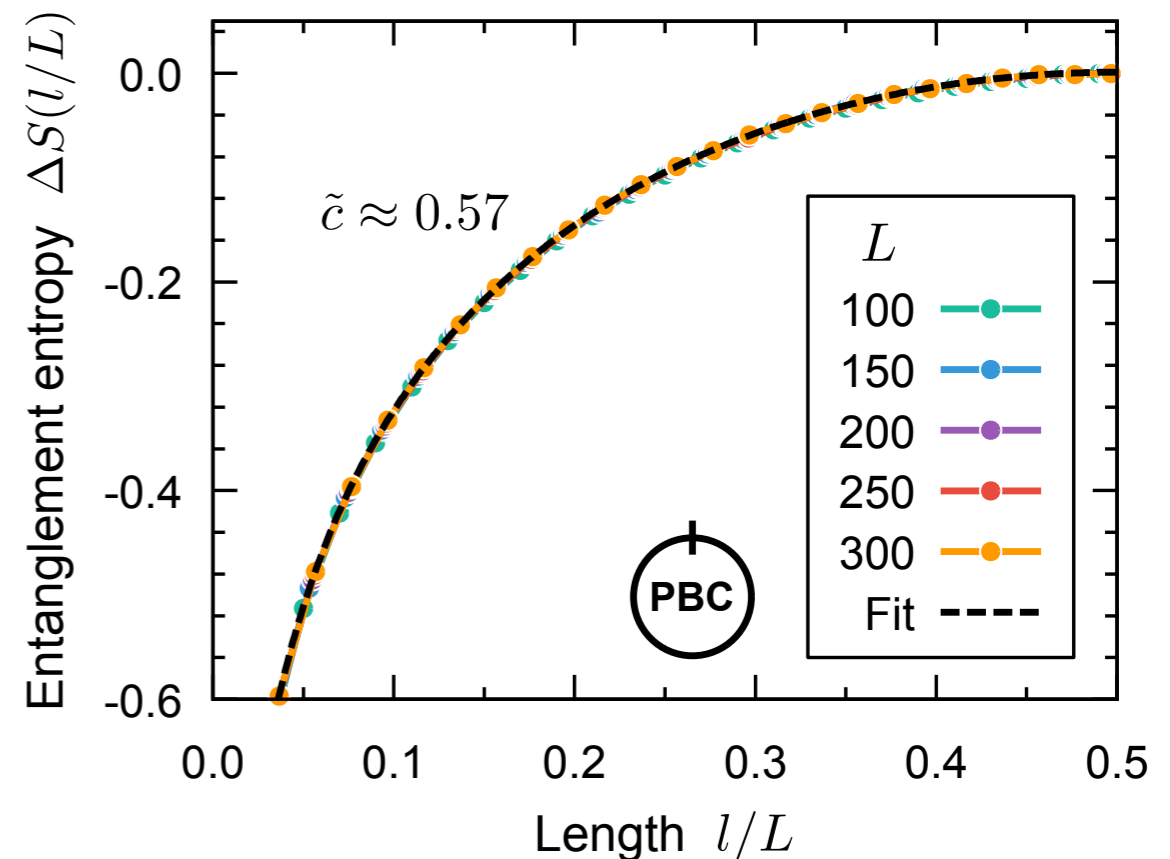
$$S_L(l) \sim \frac{c}{3} \log_2 \left[ \frac{L}{\pi} \sin \left( \pi \frac{l}{L} \right) \right]$$

$l$ : Subsystem size

$c$ : Central charge

- Normalized entanglement entropy:

$$\Delta S(l/L) \equiv S_L(l) - S_L(L/2) \sim \frac{\tilde{c}}{3} \log_2 \left[ \sin \left( \pi \frac{l}{L} \right) \right]$$



→ The PTIM satisfies the CFT-scaling!

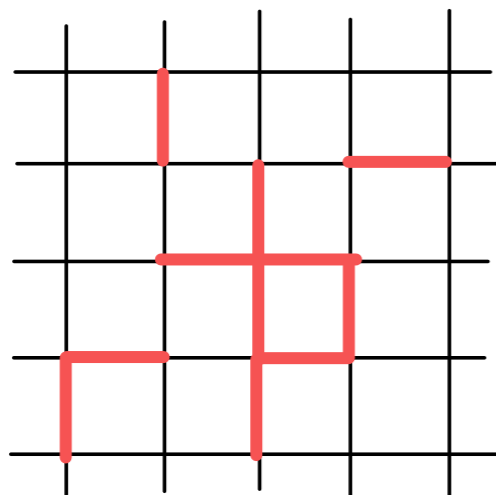
→ „Central charge“:  $\tilde{c} \approx 0.57$

**Why?**

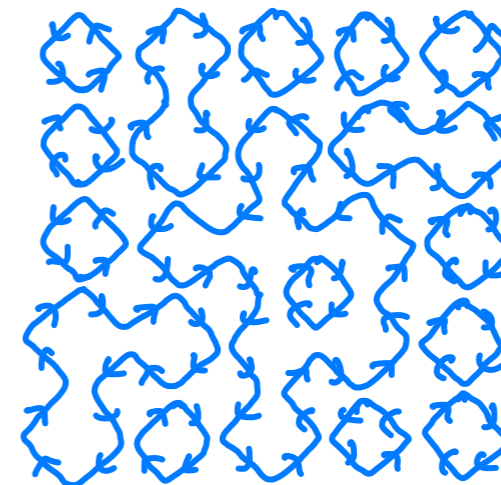
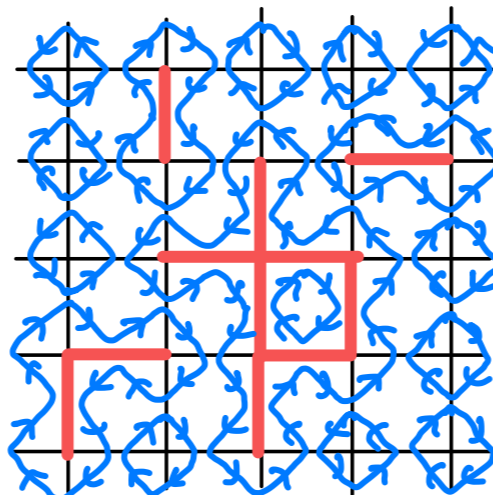
<sup>†</sup> P. Calabrese and J. Cardy, *Journal of Statistical Mechanics: Theory and Experiment* P06002 (2004)

# CFT: Model hopping

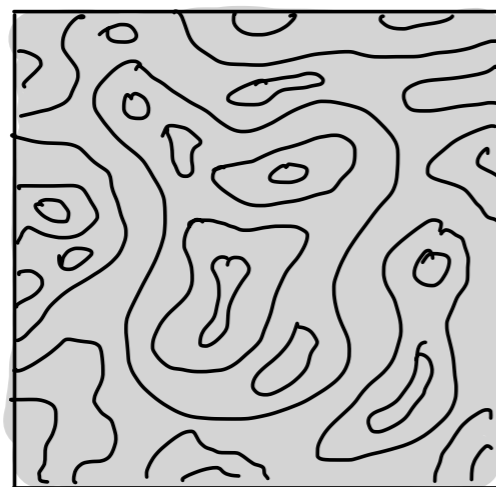
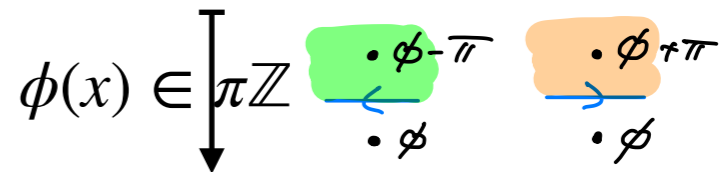
Bond percolation  
= Random cluster model



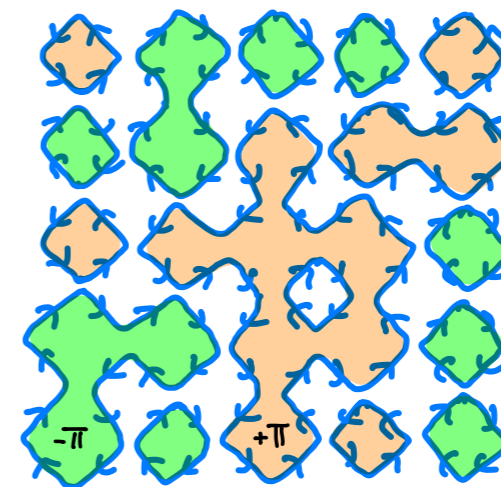
Dense gas of oriented loops



Loops = Contour lines of discrete scalar field  $\phi(x) \in \pi\mathbb{Z}$



Continuum limit



CFT = Gaussian field  $\Phi(x) \in \mathbb{R}$

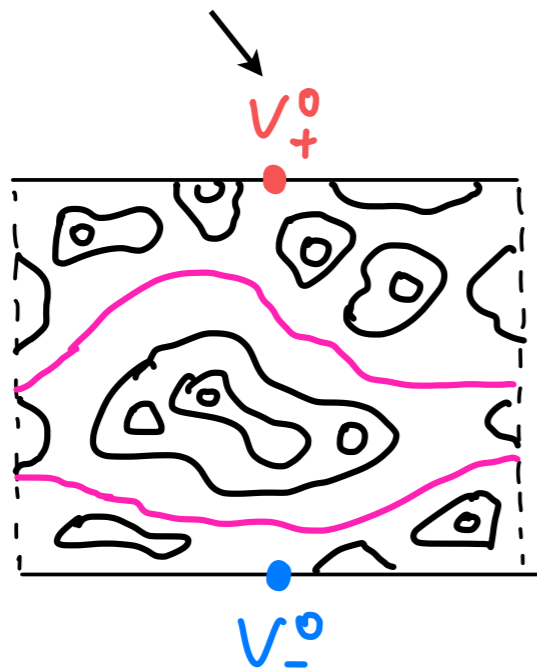
( + boundary charges )

Solid-on-solid model  $\phi(x) \in \pi\mathbb{Z}$

# CFT: Counting loops

Vertex operators

$$V_{\pm}^0 = \exp[\pm ie_0 \Phi]$$

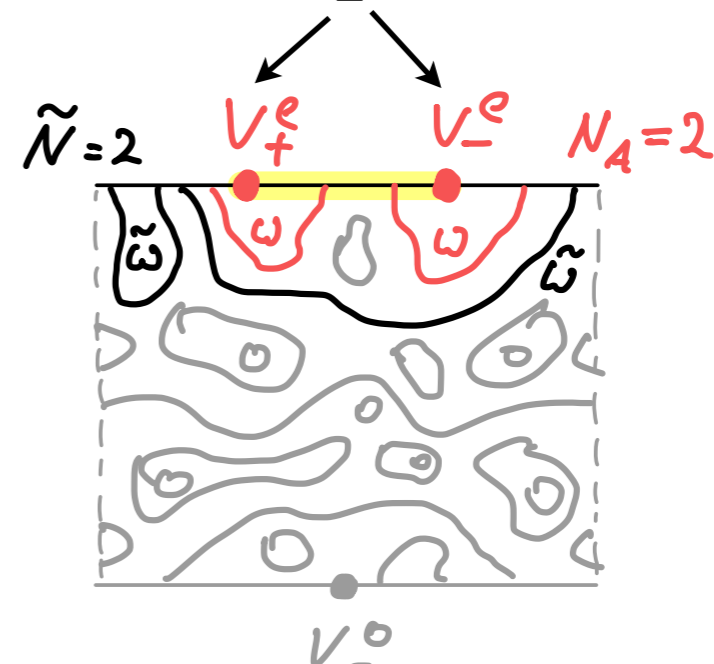


Split top charge



Vertex operators

$$V_{\pm}^e = \exp[i(\pm e + e_0/2)\Phi]$$



-> Two-point correlator = Generating function

$$\mathcal{V}_A(w) = \langle V_+^e(x_1) V_-^e(x_2) \rangle = \frac{\sum \tilde{w}^{\tilde{N}} w^{N_A}}{\sum \tilde{w}^{\tilde{N}+N_A}} \sim \frac{1}{\Delta x^{2h}}$$

**CFT !**

$$h = \frac{e^2 - (e_0/2)^2}{1 - e_0}$$

Scaling dimension†

**Examples:**  $\langle N_A \rangle = \tilde{w} [\partial_w \mathcal{V}_A(w)]_{w=\tilde{w}}$  and  $\langle \delta_{N_A=0} \rangle = \mathcal{V}_A(w \rightarrow 0)$

*Loop weights:*

$$\tilde{w} = 2 \cos(\pi e_0/2)$$

$$w = 2 \cos(\pi e)$$

*Central charge:*

$$c = 1 - 6e_0^2/(1 - e_0)$$

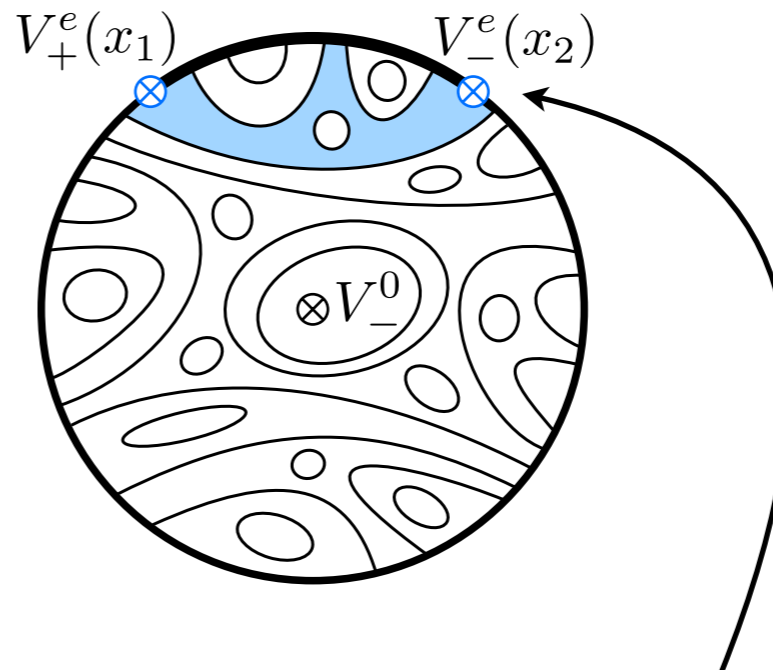
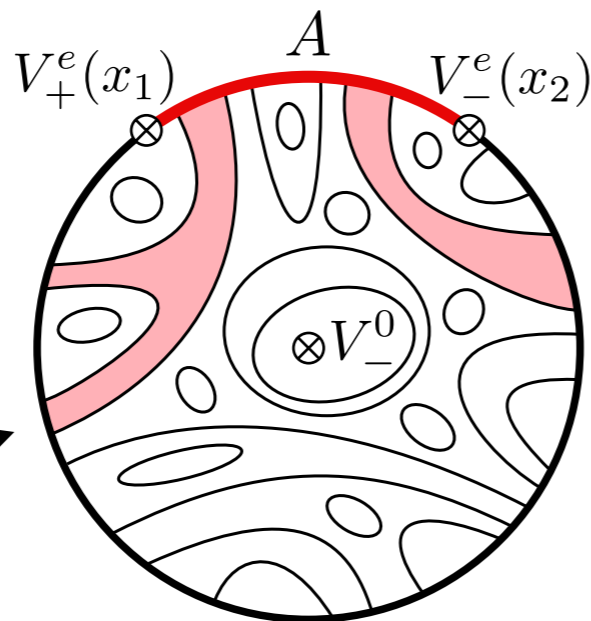
*For percolation:*

$$e_0 = 1/3 \rightarrow c = 0$$

† J. L. Jacobsen and H. Saleur, Phys. Rev. Lett. 100, 087205 (2008)

# CFT: Entanglement

Entanglement entropy  $S(A = [x_1, x_2])$  Mutual information  $I(x_1, x_2)$



- Count arcs that connect  $A$  with  $\bar{A}$   $\rightarrow N_A$
- Probability that  $x_1$  and  $x_2$  are connected ( $\Leftrightarrow N_A = 0$ )

- One finds

$$S(A) \sim \frac{\langle N_A \rangle}{2} \sim \frac{\sqrt{3} \ln 2}{2\pi} \log_2 \Delta x$$

- One finds

$$I(x_1, x_2) \sim \langle \delta_{N_A=0} \rangle \sim \frac{1}{|x_1 - x_2|^{2/3}}$$

- Therefore

$$\tilde{c} = \frac{3\sqrt{3} \ln 2}{2\pi} = 0.573\dots$$

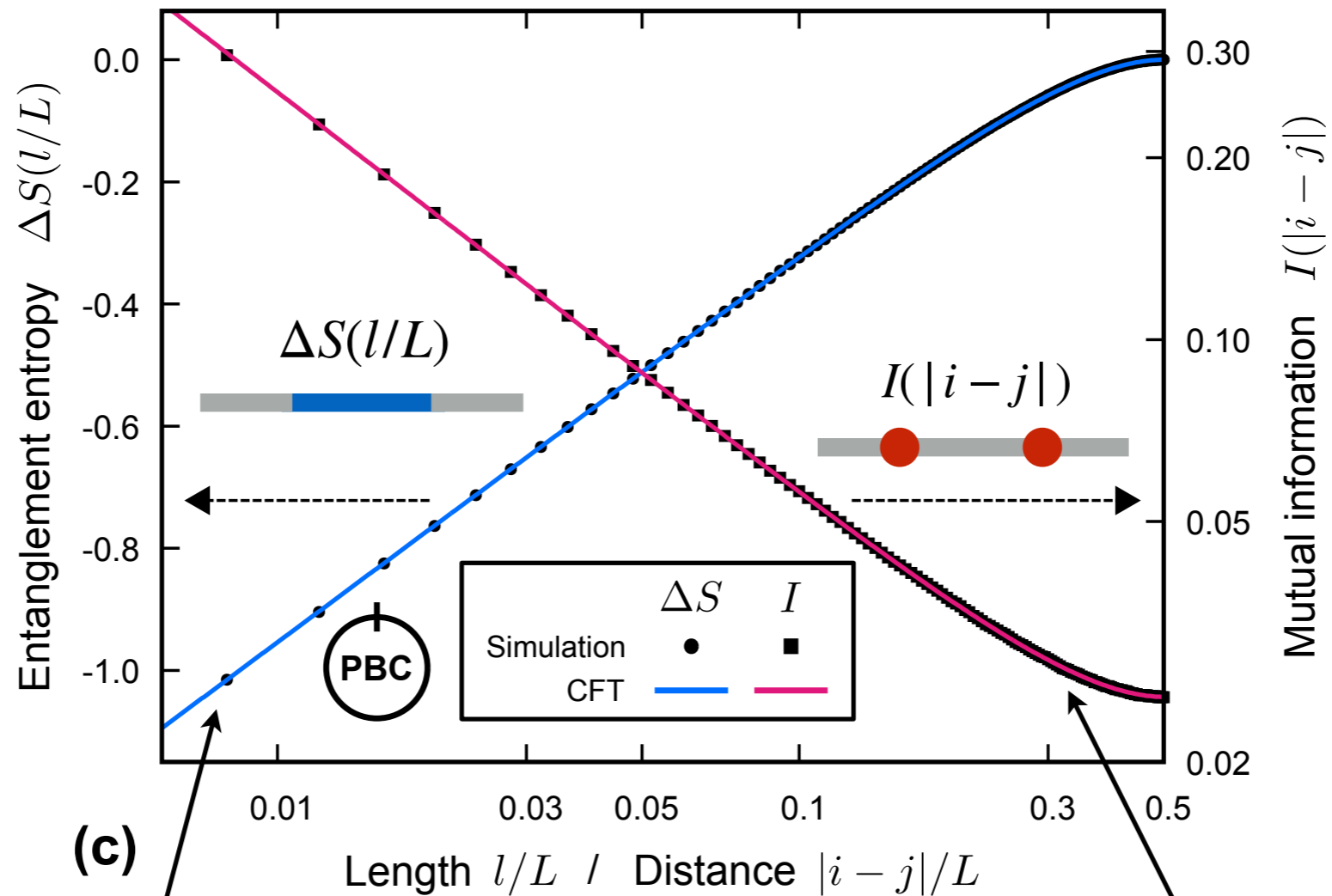


- Therefore

$$\kappa = \frac{2}{3} = 0.666\dots$$

# CFT: Numerics

Numerical results for  $L = 500$  and  $M = 10^7$  samples:



$$\Delta S(l/L) = \tilde{c}/3 \log_2 \xi(l/L) + \alpha$$

$$I(i, j) = I(|i - j|) = \beta \xi^{-\kappa}(|i - j|) + \gamma$$

$$\tilde{c} = 3\sqrt{3} \ln 2 / (2\pi)$$

$$\kappa = 2/3$$

$$\xi(x) = \sin(\pi x)$$

# EPILOGUE

# Quantum Error Correction (QEC)

## Majorana chain

- Fermionic, quadratic 1D Hamiltonian:

$$H = - \sum_e S_e = - \sum_{i=1}^{L-1} i\gamma_{2i}\gamma_{2i+1}$$

Majorana Fermions

- Two-fold degenerate ground state  
-> Quantum code:  $|\Phi\rangle = \alpha |g_0\rangle + \beta |g_1\rangle$

- Allowed **projective errors**:

$$E_i = i\gamma_{2i-1}\gamma_{2i}$$

- Allowed **syndrome measurements**:

$$S_e = i\gamma_{2i}\gamma_{2i+1}$$

### Jordan-Wigner Transformation

$$\gamma_{2i-1} = \prod_{j<i} \sigma_j^x \cdot \sigma_i^z$$

$$\gamma_{2i} = \prod_{j<i} \sigma_j^x \cdot \sigma_i^y$$

## Projective transverse field Ising model

- Ferromagnetic 1D Ising model:

$$H = - \sum_e S_e = - \sum_{i=1}^{L-1} \sigma_i^z \sigma_{i+1}^z$$

Spin-1/2

- Two-fold degenerate ground state:  
 $|\Phi\rangle = \alpha | \uparrow \dots \uparrow \rangle + \beta | \downarrow \dots \downarrow \rangle$

- Projective measurements** of

$$E_i = \sigma_i^x$$

- Measurement of **domain walls**

$$S_e = \sigma_i^z \sigma_{i+1}^z$$

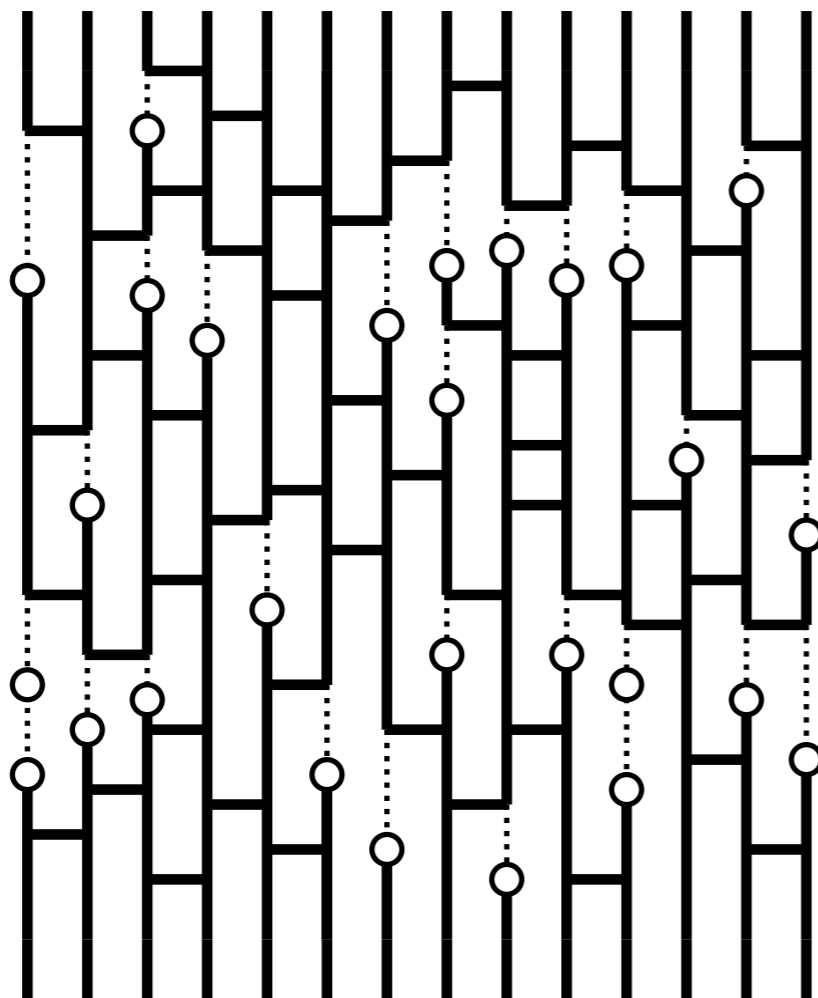
-> Survival of encoded qubit = Survival of Bell cluster amplitudes

# Spacetime Percolation and QEC

**Amplitudes preserved**

*Unknown pattern*

$$\alpha|\mathbf{m}_1\rangle + \beta|\bar{\mathbf{m}}_1\rangle$$

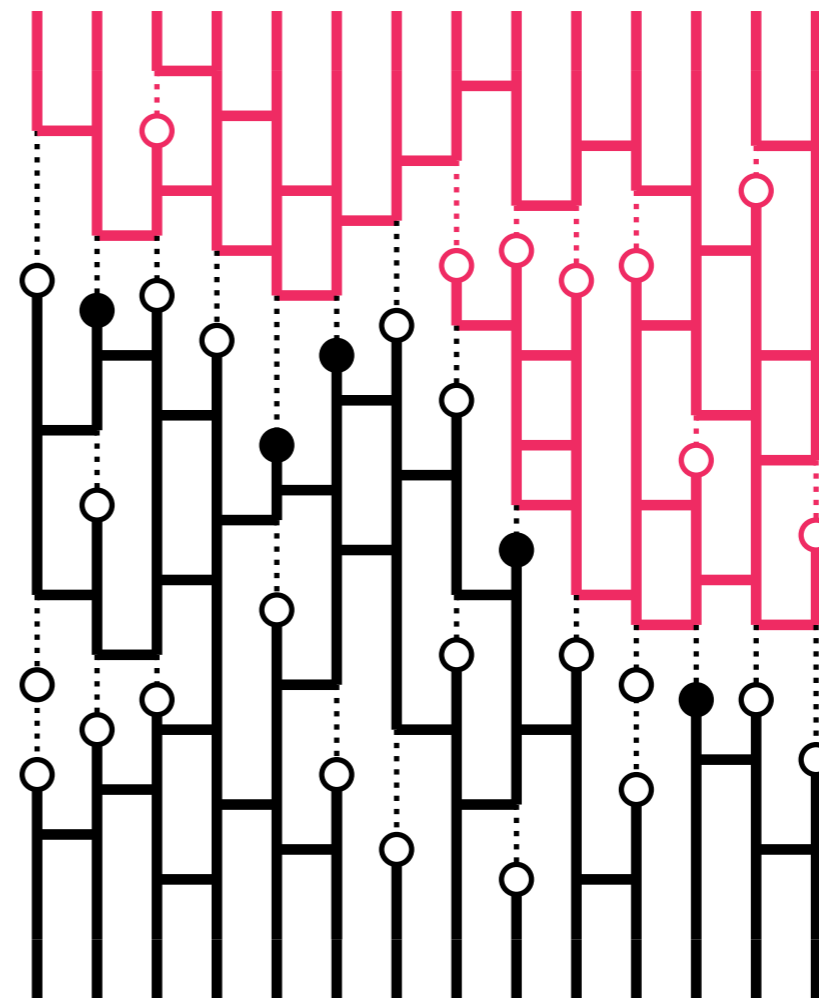


$$\alpha|0\rangle + \beta|1\rangle$$

**Amplitudes lost**

$$|\mathbf{m}_2\rangle \pm |\bar{\mathbf{m}}_2\rangle$$

$\bar{m} = \text{Complement of } m$



$$\alpha|0\rangle + \beta|1\rangle$$

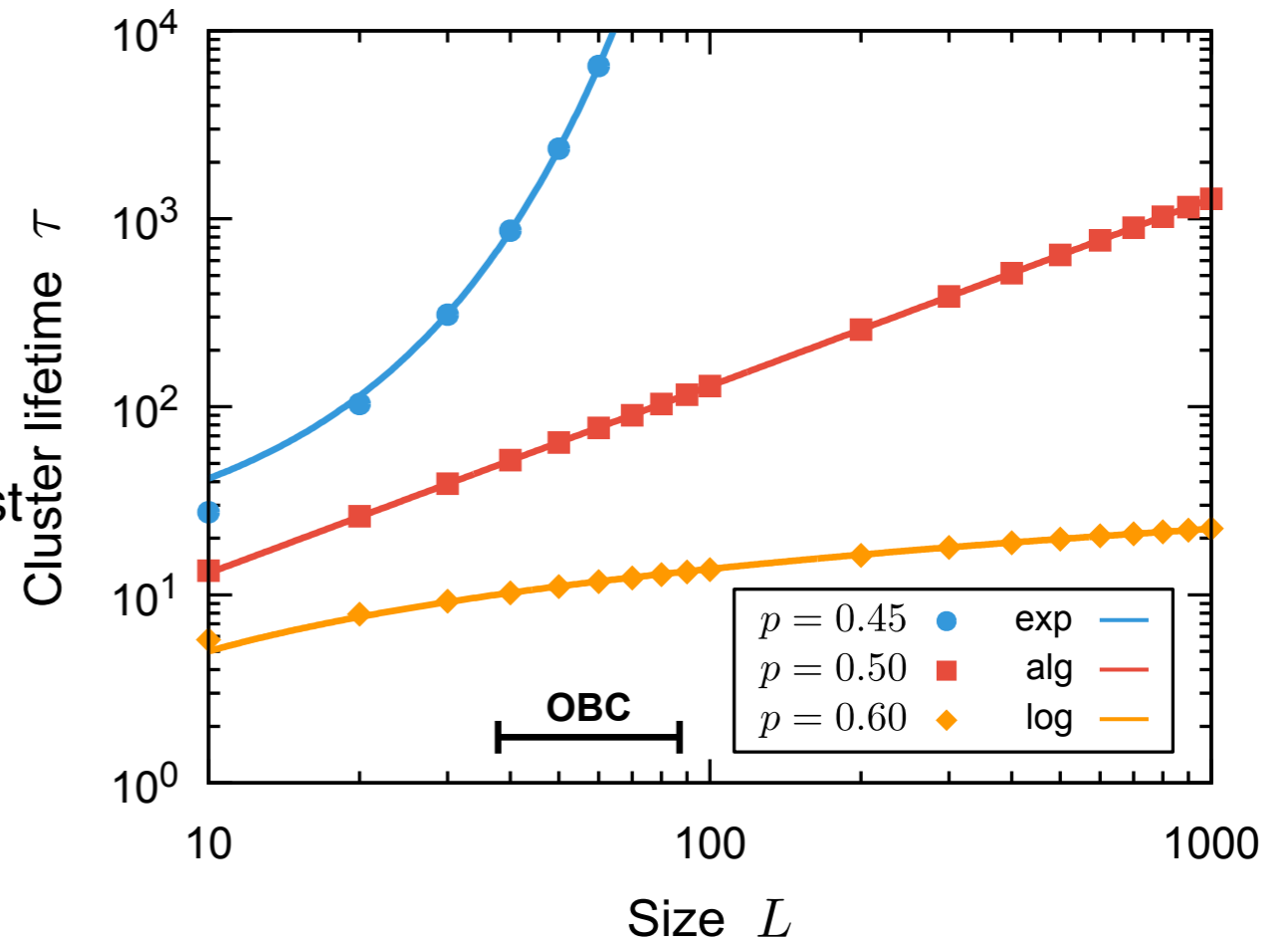
# Cluster Lifetime

## Procedure

1. Start in global Bell cluster:  
●●●●●●●●●●●●●●●●
2. Evolve with PTIM rules
3. Check each time step if a Bell cluster with color ● exists
4. Lifetime  $\tau$  = Time after which color ● is lost

## Results

- $p < p_c$ : Exponential growth
- $p = p_c = 0.5$ : Algebraic ( $\sim$ linear) growth
- $p > p_c$ : Logarithmic growth

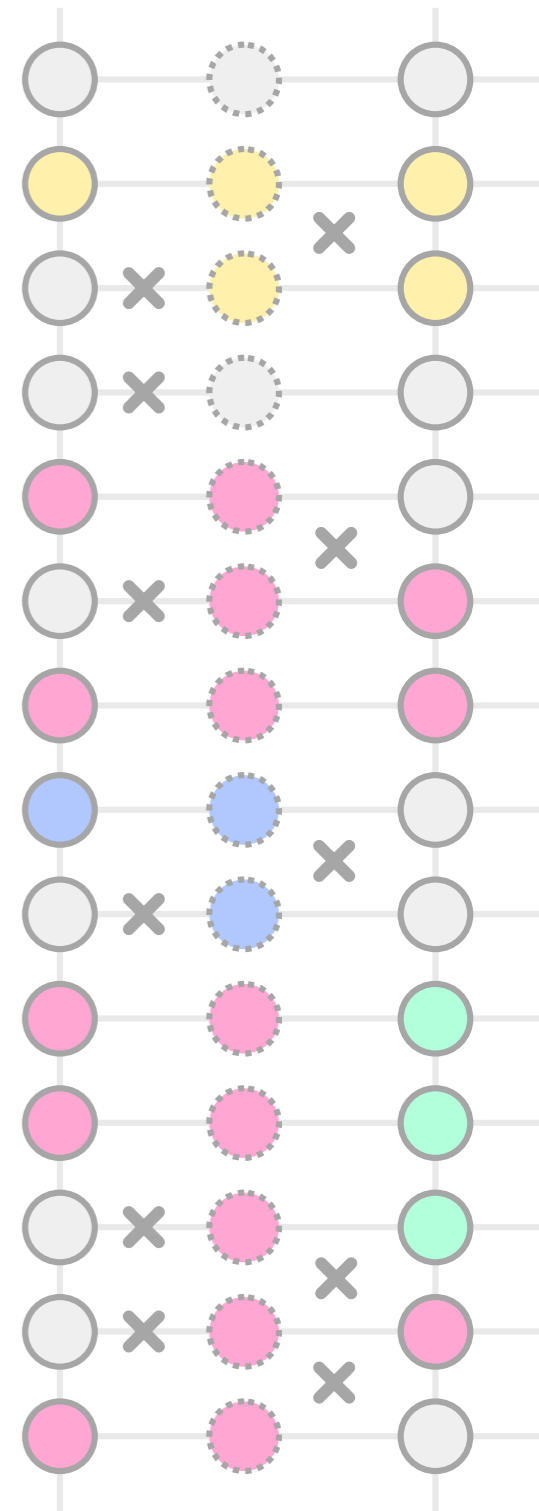


-> For  $p < p_c$  amplitudes are preserved by measurements of stabilizers !

**Open question:** When and how can the amplitudes be accessed ( = decoding )

# Summary

- **Projective transverse field Ising model**  
= competition of **non-commuting, projective measurements**
- Entanglement described by the **Colored Cluster Model**
- Can be **efficiently simulated** (stabilizer and CCM)
- **Continuous entanglement transition:**  
 $p < p_c$ : long-range entangled (global Bell cluster)  
 $p > p_c$ : short-range entangled (local Bell clusters)
- Dynamic phase transition = **Spacetime percolation** of Bell clusters
- Critical behaviour can be described by **conformal field theory**
- Can be interpreted as competition between **projective errors** and **stabilizer measurements** of a quantum code

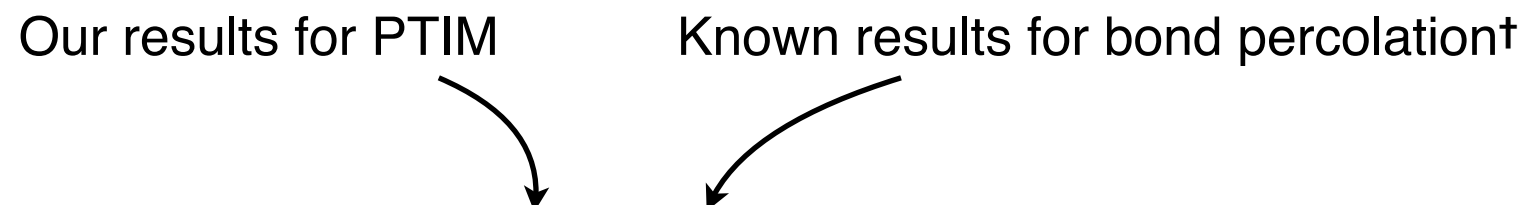


# Beyond one dimension

PTIM on lattice  $\mathcal{L}$  = Bond percolation on  $\mathcal{L} \times \mathbb{Z}$

Our results for PTIM

Known results for bond percolation†



Dimension	Lattice	$p_c$	$\tilde{p}_c$	Percolation lattice
1	-	0.5	0.5*	Square
2	Square	0.75	0.7512	Cubic
2	Kagome	0.74	0.7437	stacked Kagome
2	Honeycomb	0.70	0.6907	stacked Honeycomb
2	Triangular	0.83	0.8140	stacked Triangular

† S. C. van der Marck, *Percolation thresholds and universal formulas*, Phys. Rev. E 55, 1514 (1997)