

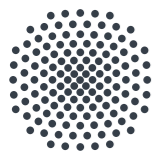
A pedagogical introduction to the

Classification of one-dimensional Symmetry Protected Topological Phases

Nicolai Lang

Status Workshop
Institute for Theoretical Physics III
University of Stuttgart

September 6th – 9th 2018
Zernez, Switzerland



University of Stuttgart
Germany



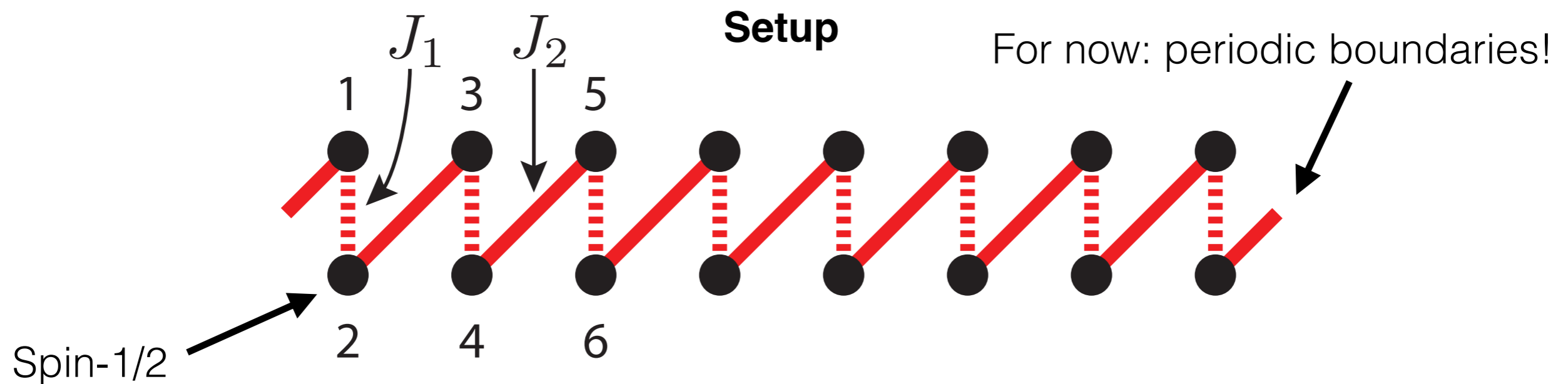
European Research Council
Established by the European Commission

Theory

Application

Motivation

(Details → Sebastian Webers talk)



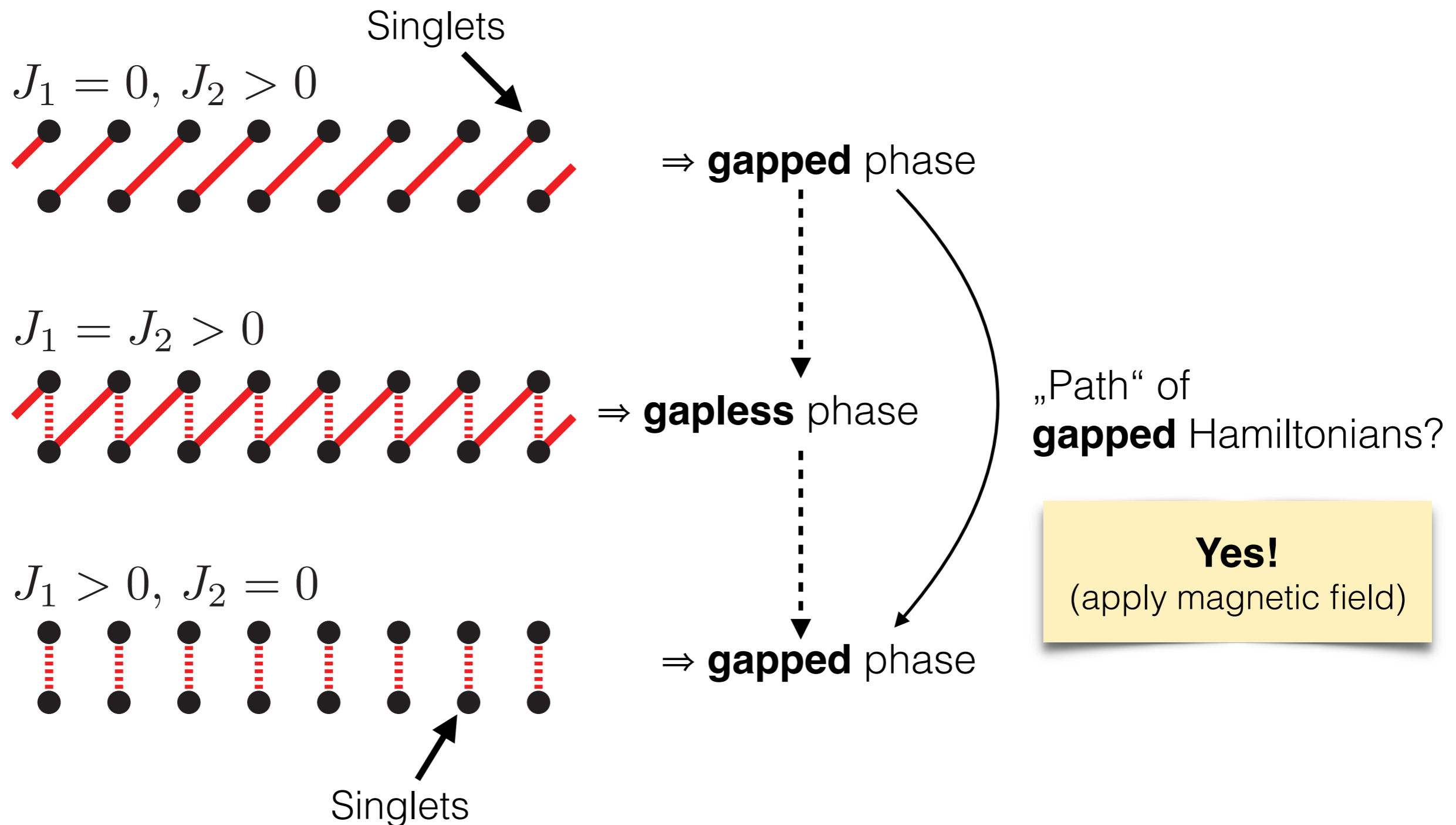
Hamiltonian

$$\begin{aligned}
 H &= J_1 \sum_{i=1}^L (\sigma_{2i-1}^- \sigma_{2i}^+ + \sigma_{2i}^- \sigma_{2i-1}^+) + J_2 \sum_{i=1}^L (\sigma_{2i}^- \sigma_{2i+1}^+ + \sigma_{2i+1}^- \sigma_{2i}^+) \\
 &= \frac{J_1}{2} \sum_{i=1}^L (\sigma_{2i-1}^x \sigma_{2i}^x + \sigma_{2i-1}^y \sigma_{2i}^y) + \frac{J_2}{2} \sum_{i=1}^L (\sigma_{2i}^x \sigma_{2i+1}^x + \sigma_{2i}^y \sigma_{2i+1}^y)
 \end{aligned}$$

(Jordan-Wigner transformation of fermionic **SSH chain**)

Motivation

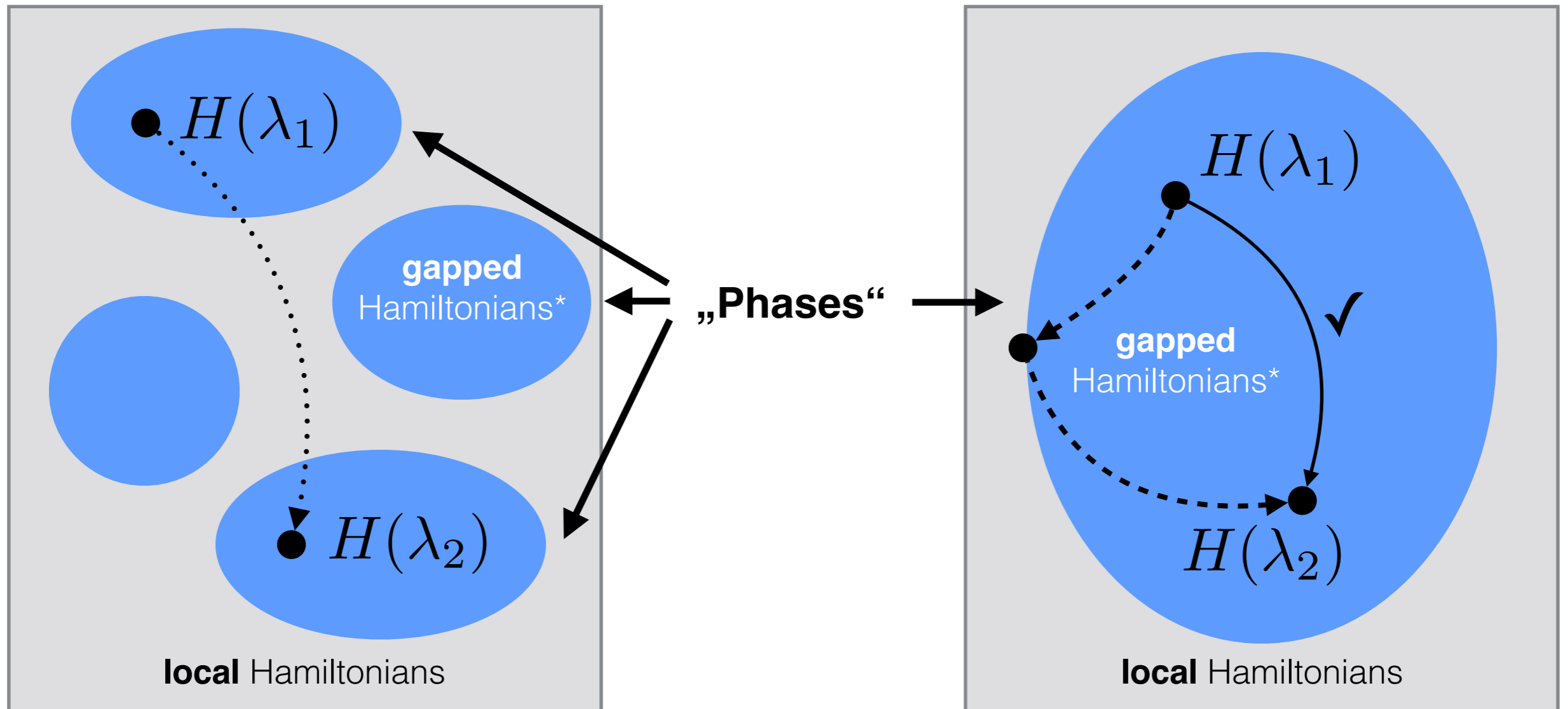
Ground state phases?



Topological Phases

$$D \geq 2$$

$$D = 1$$



Many
(topological) phases

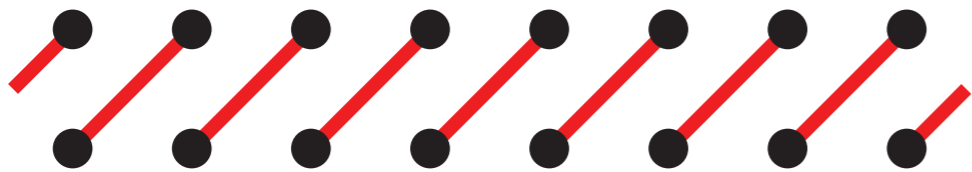
* = with unique ground states
(no symmetry breaking)

Only **one**
(trivial) phase

Motivation

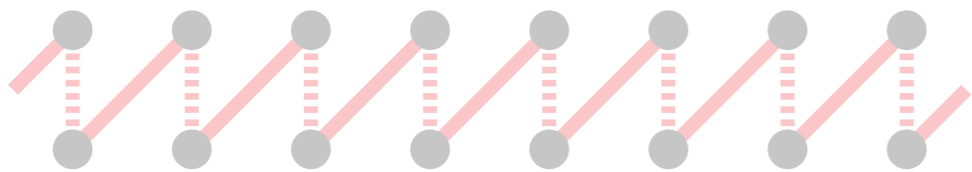
Ground state phases?

$$J_1 = 0, J_2 > 0$$



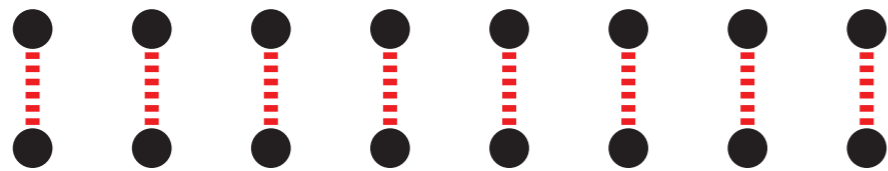
⇒ **gapped** phase

$$J_1 = J_2 > 0$$



⇒ **gapless** phase

$$J_1 > 0, J_2 = 0$$



⇒ **gapped** phase

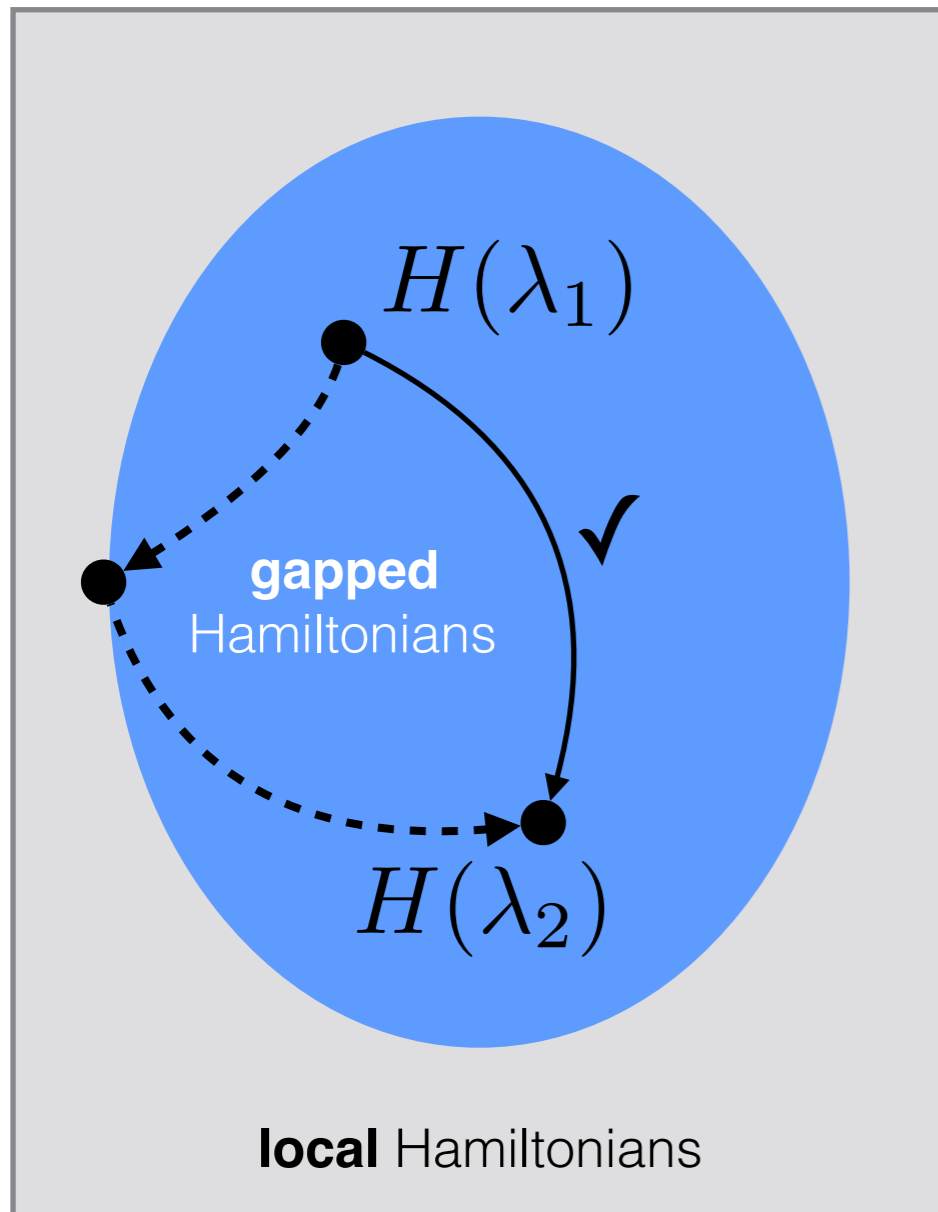
The **same** phase!

Use a universal recipe:

Trivial concept + Additional **constraints** = **Non-trivial** concept

Use a universal recipe:

Trivial concept + Additional **constraints** = **Non-trivial** concept

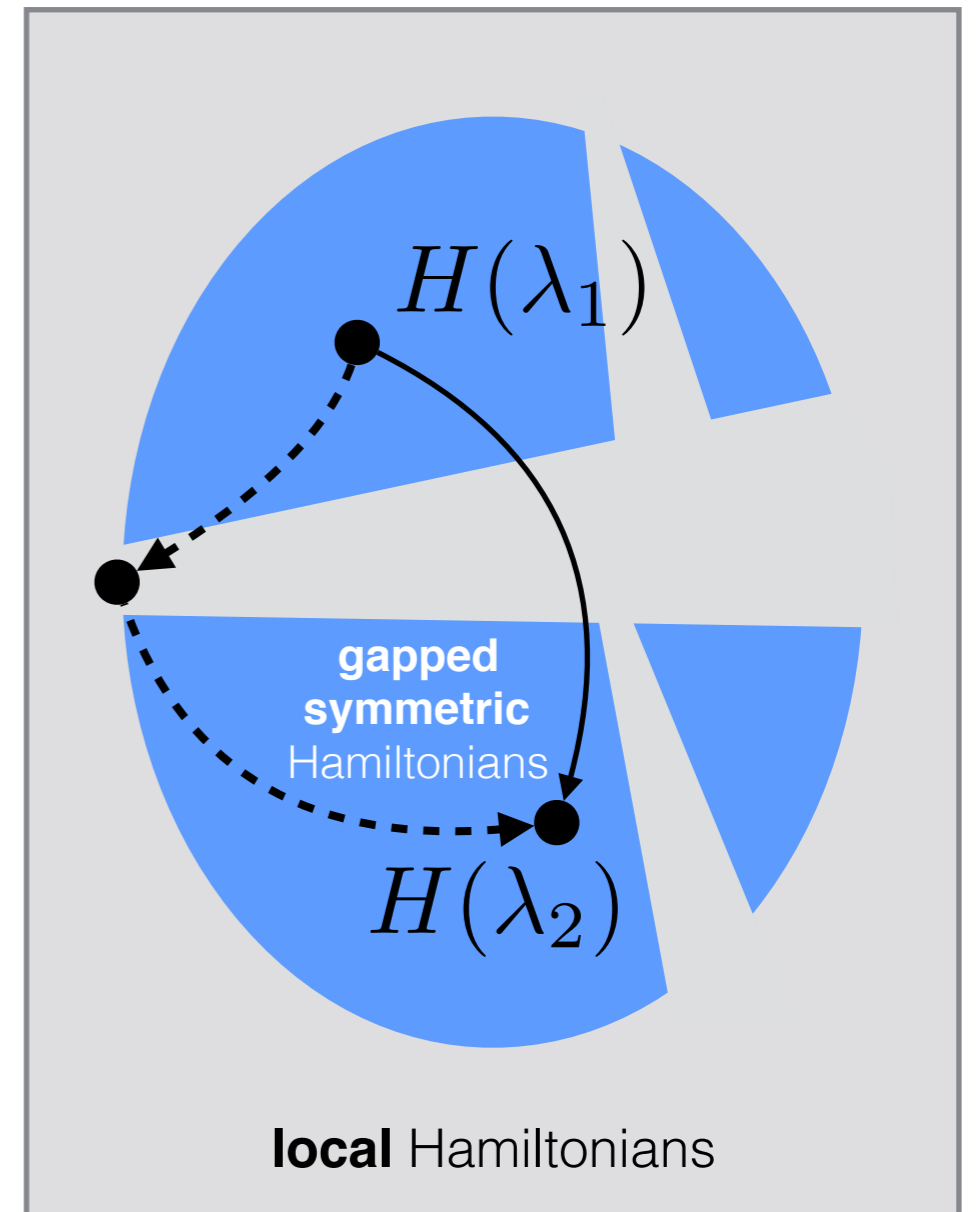


(Intrinsic)
topological phases

Require
symmetry
 S

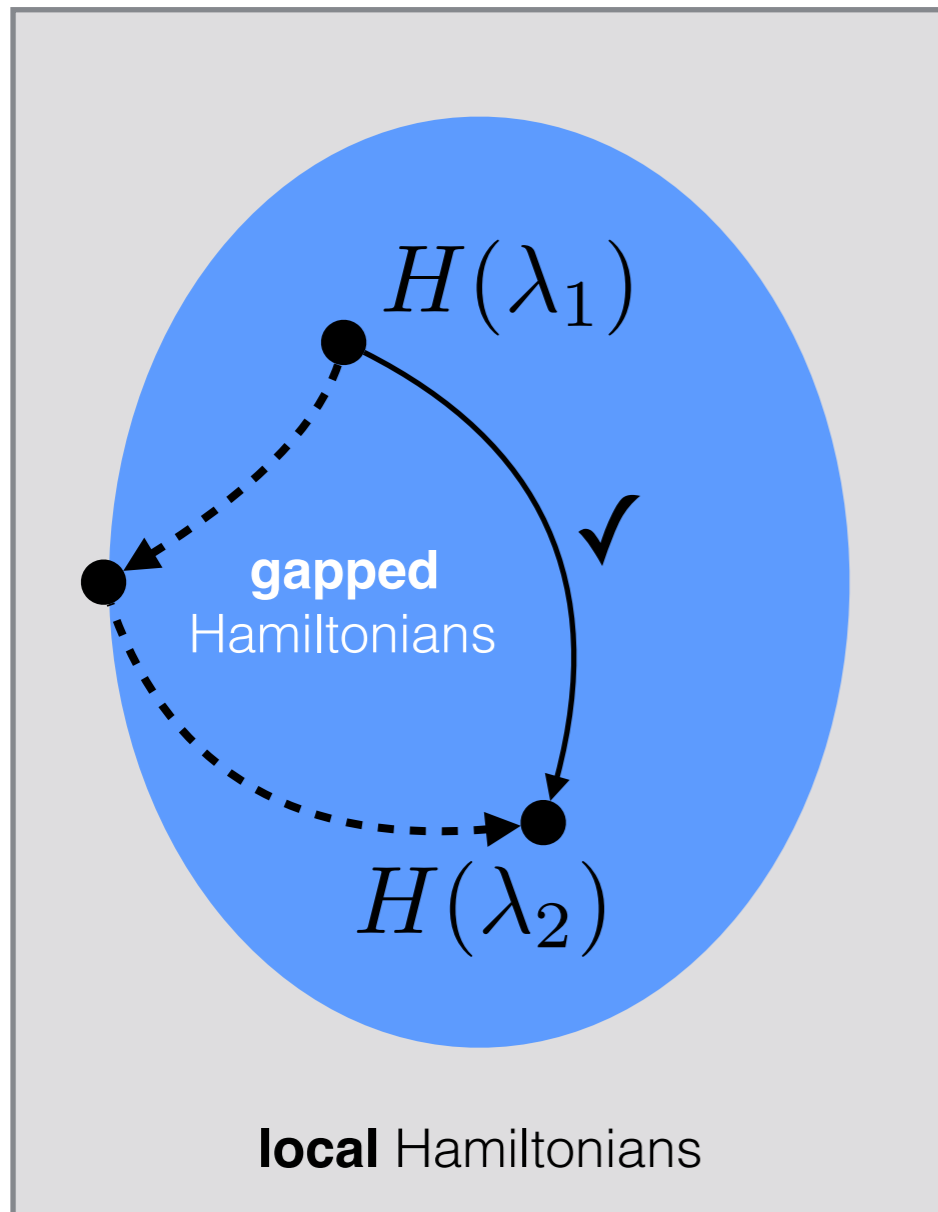
→

$[H, S] = 0$

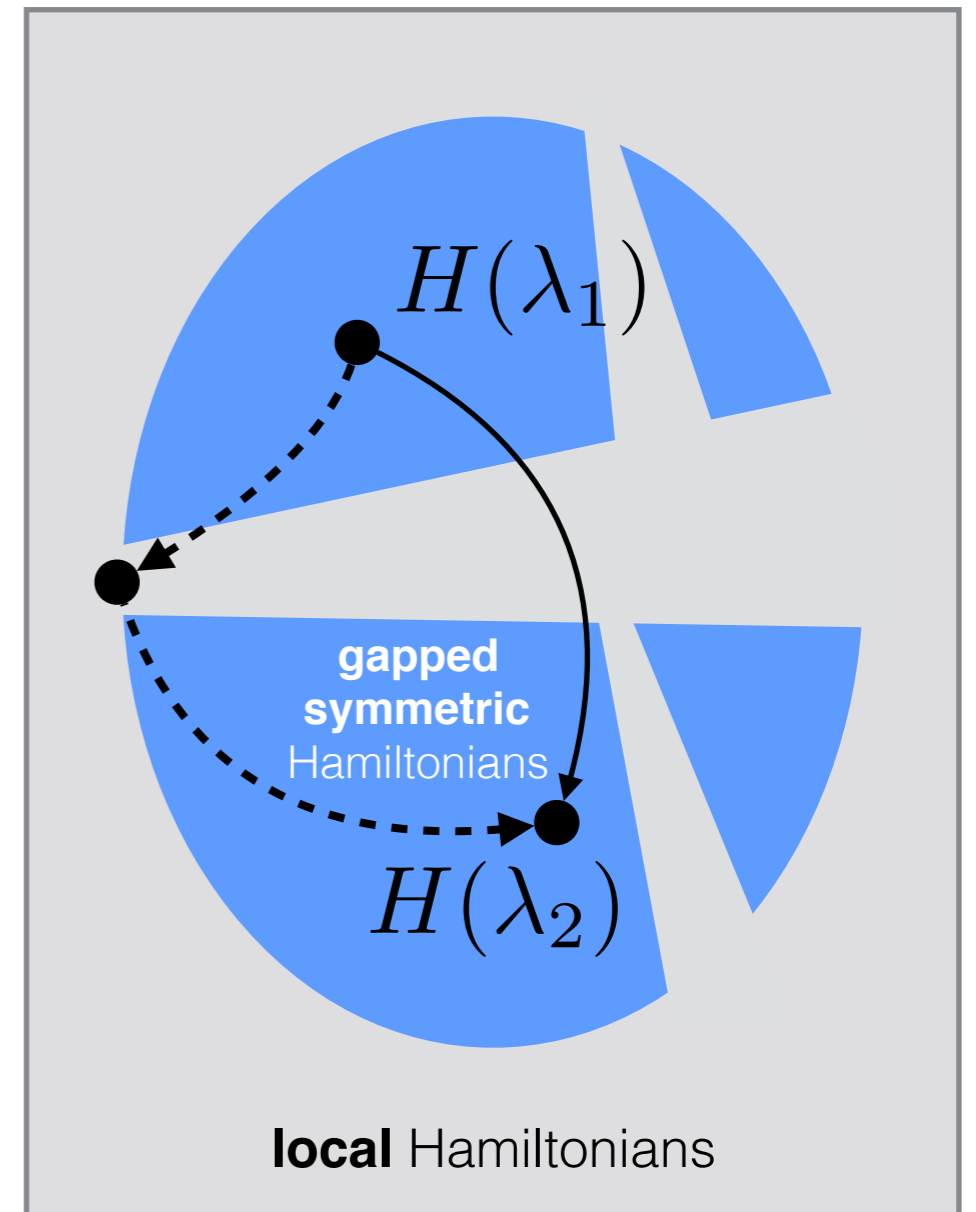
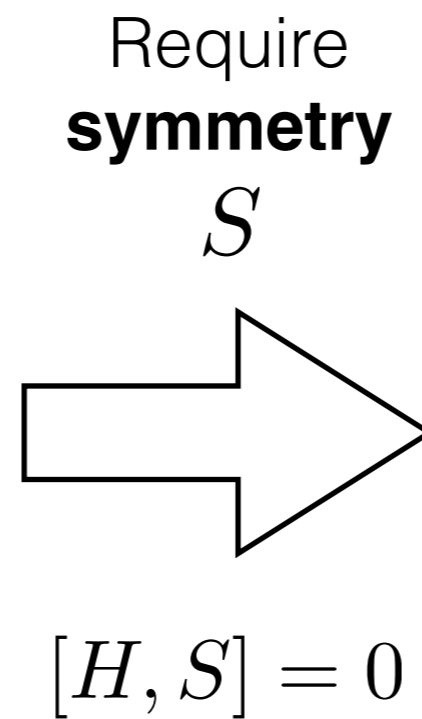


Symmetry-protected
topological phases

SPT Phases



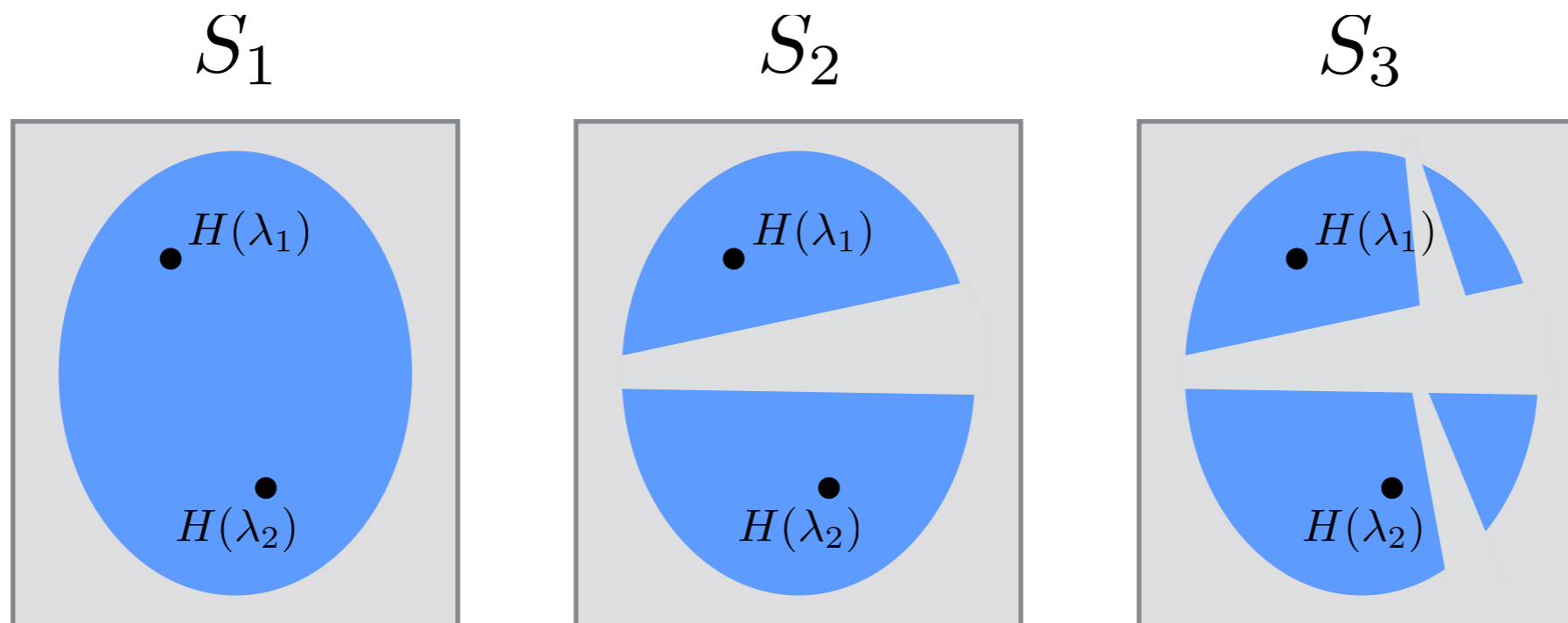
(Intrinsic)
topological phases



Symmetry-protected
topological phases

SPT Phases

„Splitting“ depends on the symmetry (representation) \mathcal{S} :



Given a symmetry, how to **classify** the possible SPT phases?

Symmetries

Hamiltonian

$$\frac{J_1}{2} \sum_{i=1}^L (\sigma_{2i-1}^x \sigma_{2i}^x + \sigma_{2i-1}^y \sigma_{2i}^y) + \frac{J_2}{2} \sum_{i=1}^L (\sigma_{2i}^x \sigma_{2i+1}^x + \sigma_{2i}^y \sigma_{2i+1}^y)$$

Symmetries

$$\textit{Particle number: } R_\phi = \exp \left[i\phi \sum_i \frac{1 - \sigma_i^z}{2} \right] \rightarrow \text{U}(1)$$

$$\textit{Parity: } Z = \prod_i \sigma_i^z \rightarrow \mathbb{Z}_2$$

$$\textit{Time-reversal: } T = K \rightarrow \mathbb{Z}_2^T$$

$$\textit{Particle-hole: } X = \prod_i \sigma_i^x \rightarrow \mathbb{Z}_2$$

$$\textit{Chiral: } XK = \prod_i \sigma_i^x \circ K \rightarrow \mathbb{Z}_2^T$$

Symmetries

Hamiltonian

$$\frac{J_1}{2} \sum_{i=1}^L (\sigma_{2i-1}^x \sigma_{2i}^x + \sigma_{2i-1}^y \sigma_{2i}^y) + \frac{J_2}{2} \sum_{i=1}^L (\sigma_{2i}^x \sigma_{2i+1}^x + \sigma_{2i}^y \sigma_{2i+1}^y)$$

Symmetries

Particle number: $R_\phi = \exp \left[i\phi \sum_i \frac{1 - \sigma_i^z}{2} \right] \rightarrow \text{U}(1)$

Parity: $Z = \prod_i \sigma_i^z \rightarrow \mathbb{Z}_2$

Time-reversal: $T = K \rightarrow \mathbb{Z}_2^T$

Particle-hole: $X = \prod_i \sigma_i^x \rightarrow \mathbb{Z}_2$

Chiral: $XK = \prod_i \sigma_i^x \circ K \rightarrow \mathbb{Z}_2^T$

Experiment: $\text{U}(1) \times \mathbb{Z}_2^T$
(Sebastians talk)

Symmetries

Hamiltonian

$$\frac{J_1}{2} \sum_{i=1}^L (\sigma_{2i-1}^x \sigma_{2i}^x + \sigma_{2i-1}^y \sigma_{2i}^y) + \frac{J_2}{2} \sum_{i=1}^L (\sigma_{2i}^x \sigma_{2i+1}^x + \sigma_{2i}^y \sigma_{2i+1}^y)$$

Symmetries

Particle number: $R_\phi = \exp \left[i\phi \sum_i \frac{1 - \sigma_i^z}{2} \right] \rightarrow \text{U}(1)$

Parity: $Z = \prod_i \sigma_i^z \rightarrow \mathbb{Z}_2$

Time-reversal: $T = K \rightarrow \mathbb{Z}_2^T$

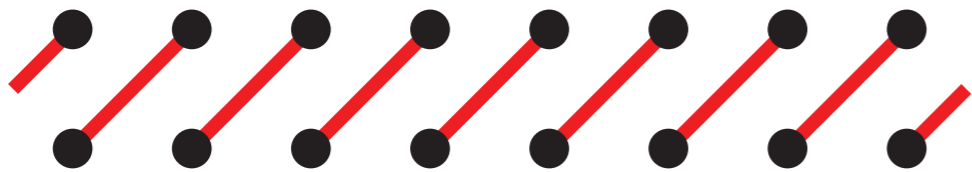
Particle-hole: $X = \prod_i \sigma_i^x \rightarrow \mathbb{Z}_2$

Chiral: $XK = \prod_i \sigma_i^x \circ K \rightarrow \mathbb{Z}_2^T$

In the following: $D_2 = \mathbb{Z}_2 \times \mathbb{Z}_2 = \langle X, Z \rangle$

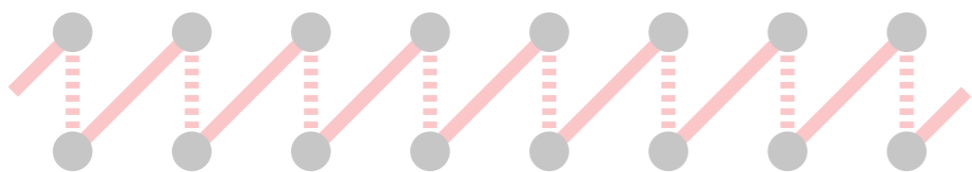
Symmetries

$$J_1 = 0, J_2 > 0$$



⇒ **gapped** phase

$$J_1 = J_2 > 0$$



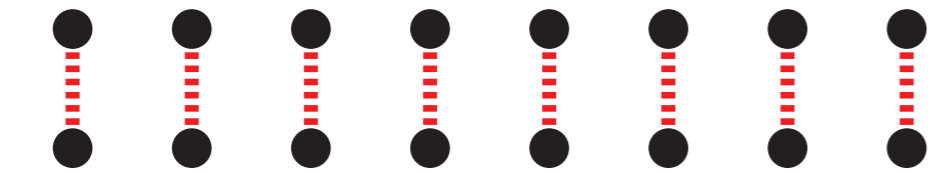
⇒ **gapless** phase

D_2 symmetric

D_2 breaking

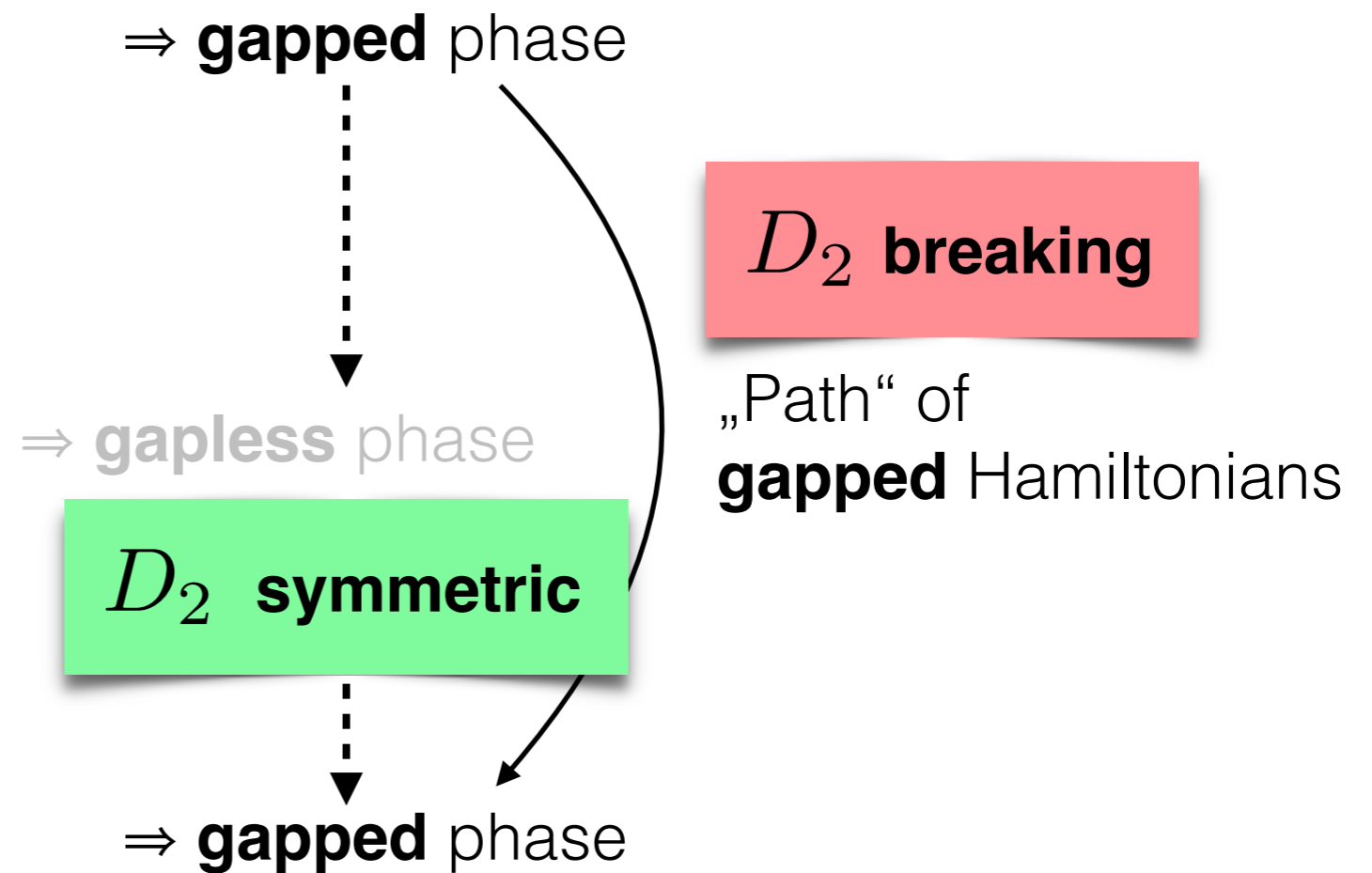
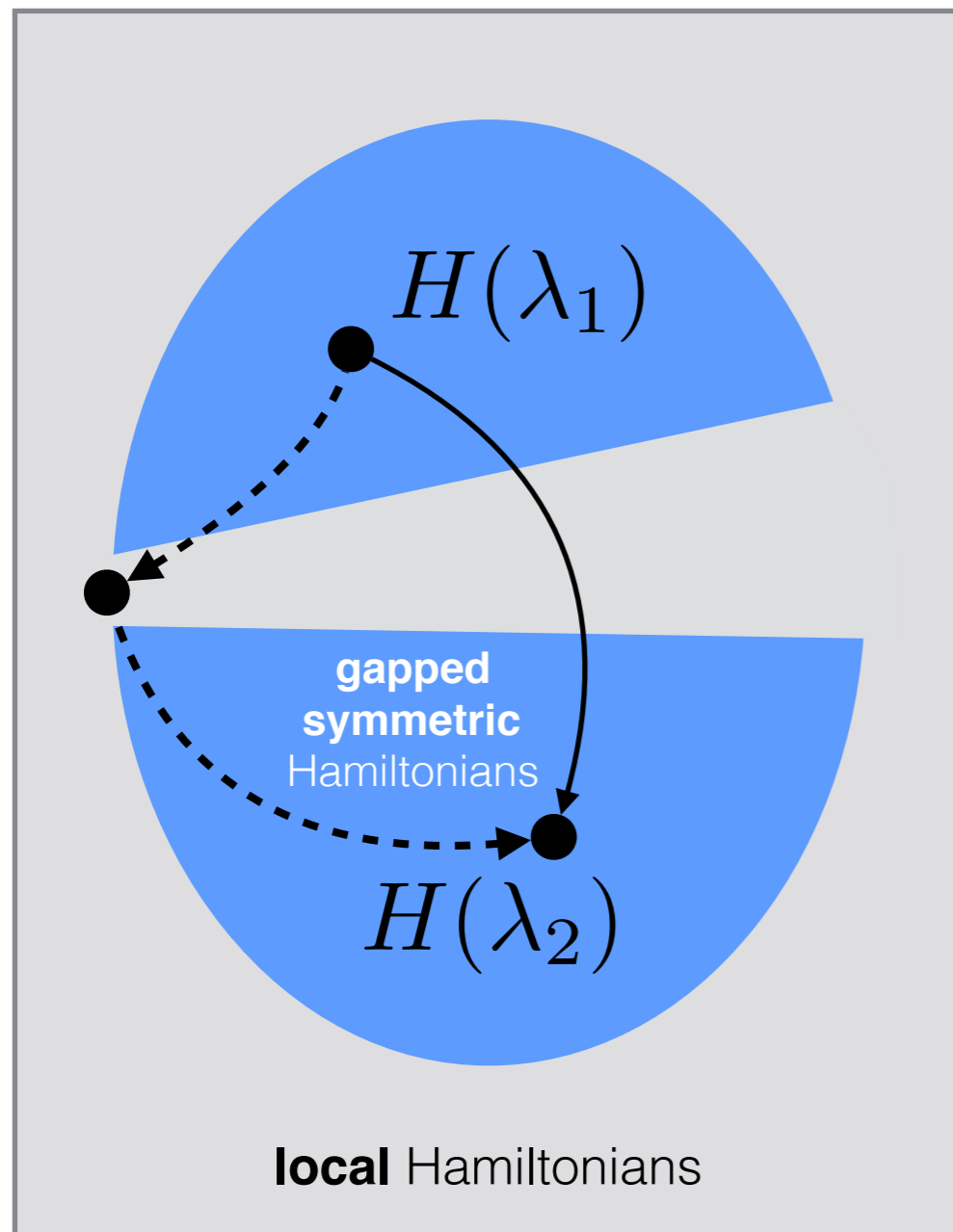
„Path“ of
gapped Hamiltonians

$$J_1 > 0, J_2 = 0$$

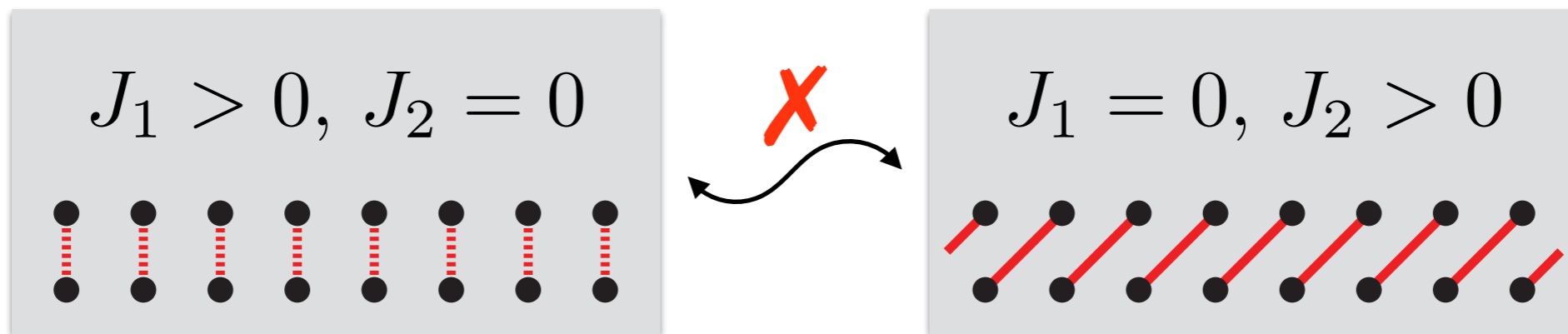


⇒ **gapped** phase

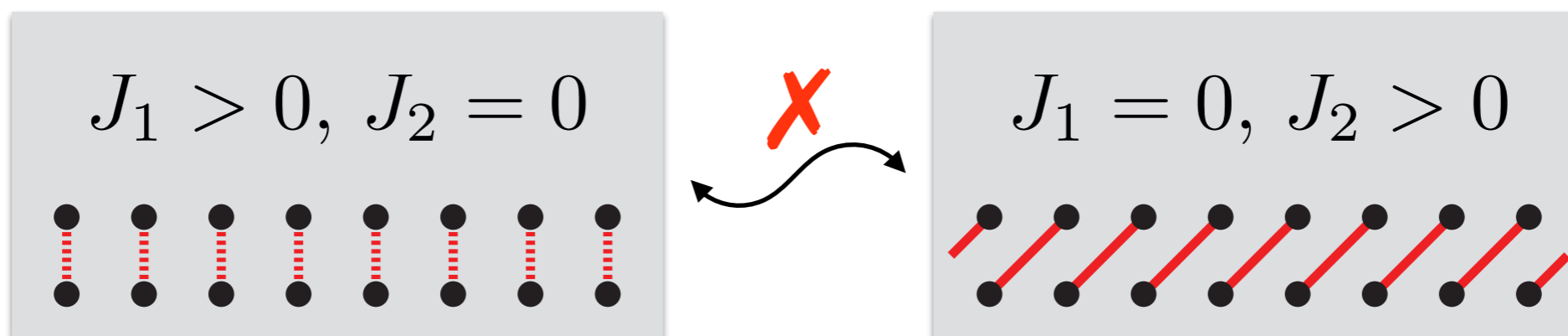
Symmetries



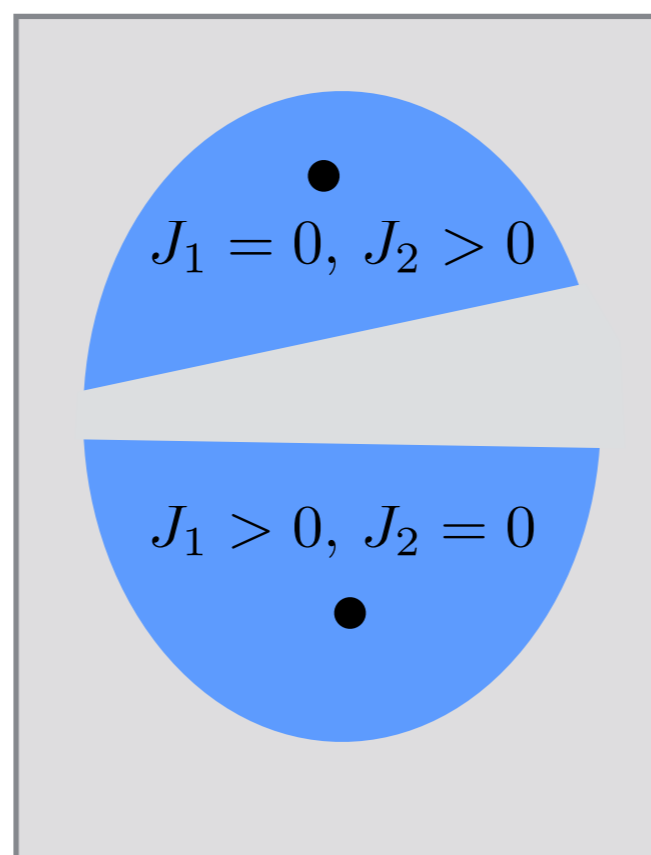
How to prove that there is **no** path of **gapped**, D_2 -**symmetric** Hamiltonians connecting the two phases?



How to prove that there is **no** path of **gapped**, D_2 -**symmetric** Hamiltonians connecting the two phases?



How to prove that it looks like this:



Matrix Product States

- Ground states of **local, gapped, one-dimensional** Hamiltonians are **short-range entangled**
- They can be efficiently encoded as **matrix-product states (MPS)**:

$$|\Omega\rangle = \sum_{\{i_k\}} \text{Tr} \left[A^{i_1} A^{i_2} \dots A^{i_{2L}} \right] |i_1, i_2, \dots, i_{2L}\rangle$$

$D \times D$ -matrices
 „Bond dimension“

„efficiently“ =
finite Bond dimension
in thermodynamic limit

- Crucial for numerics: density matrix renormalization group (DMRG)
- *Here:* Use MPS to **classify SPT phases!**

Matrix Product States

$$\begin{aligned}
 |A\rangle &= \bigotimes_{i=1}^L (|0\rangle_{2i}|1\rangle_{2i+1} + |1\rangle_{2i}|0\rangle_{2i+1}) \\
 &= \sum_{\{i_k\}} \text{Tr} \left[A^{i_1 i_2} A^{i_3 i_4} \dots A^{i_{2L-1} i_{2L}} \right] |i_1 i_2, \dots, i_{2L-1} i_{2L}\rangle
 \end{aligned}$$

$$\downarrow \\
 A_{\alpha\beta}^{ij} \equiv \left(A^{ij} \right)_{\alpha\beta} = \delta_{i\alpha} \sigma_{j\beta}^x$$

$$D = 2$$

$$J_1 = 0, J_2 < 0$$



$$J_1 < 0, J_2 = 0$$



$$\begin{aligned}
 |B\rangle &= \bigotimes_{i=1}^L (|0\rangle_{2i-1}|1\rangle_{2i} + |1\rangle_{2i-1}|0\rangle_{2i}) \\
 &= \sum_{\{i_k\}} \text{Tr} \left[B^{i_1 i_2} B^{i_3 i_4} \dots B^{i_{2L-1} i_{2L}} \right] |i_1 i_2, \dots, i_{2L-1} i_{2L}\rangle
 \end{aligned}$$

$$\downarrow \\
 B_{\alpha\beta}^{ij} \equiv \left(B^{ij} \right)_{\alpha\beta} = \sigma_{ij}^x$$

$$D = 1$$

Symmetric MPS

Representation of symmetry group G :

$$\rho(g_1)\rho(g_2) = \rho(g_1g_2) \quad \text{for } g_1, g_2 \in G$$

$$[H, \rho(g)] = 0$$

Symmetry acts **locally** on each site:

$$\rho(g) = \pi(g) \otimes \pi(g) \cdots \otimes \pi(g)$$

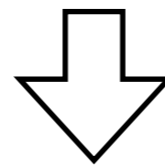


Unitary representations:

$$\pi(g_1)\pi(g_2) = \pi(g_1g_2) \quad \text{for } g_1, g_2 \in G$$

Symmetric ground state $|A\rangle$:
(no symmetry breaking)

$$\rho(g)|A\rangle = \alpha(g)|A\rangle \quad \text{with } |\alpha(g)| = 1$$



$$\rho(g)|A\rangle = \sum_{\{i_k\}} \text{Tr} [A^{i_1} \cdots A^{i_{2L}}] \rho(g)|i_1, \dots, i_{2L}\rangle$$

$$= \sum_{\{i_k\}} \text{Tr} [\tilde{A}^{i_1} \cdots \tilde{A}^{i_{2L}}] |i_1, \dots, i_{2L}\rangle \quad \text{with}$$

$$\tilde{A}^i = \sum_j [\pi(g)]_{ij} A^j$$

$$= \gamma(g) V^{-1}(g) \cdot A^i \cdot V(g)$$


phase

$D \times D$ -matrices

Symmetric MPS

Representation of symmetry group $D_2 = \mathbb{Z}_2 \times \mathbb{Z}_2 = \{1, x, z, xz\}$

$$\rho(x) = \prod_i \underbrace{\sigma_{2i-1}^x \sigma_{2i}^x}_{\pi_i(x)} = X \quad \text{and} \quad \rho(z) = \prod_i \underbrace{\sigma_{2i-1}^z \sigma_{2i}^z}_{\pi_i(z)} = Z$$

← **local** action on


Symmetric ground states:

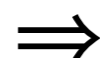
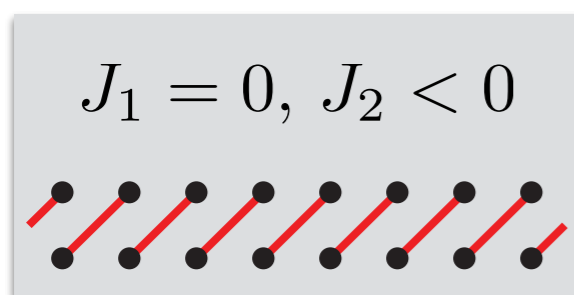
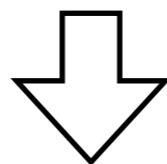
$$|A\rangle = \bigotimes_{i=1}^L (|0\rangle_{2i}|1\rangle_{2i+1} + |1\rangle_{2i}|0\rangle_{2i+1})$$

$$|B\rangle = \bigotimes_{i=1}^L (|0\rangle_{2i-1}|1\rangle_{2i} + |1\rangle_{2i-1}|0\rangle_{2i})$$

$$X|A, B\rangle = |A, B\rangle$$

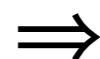
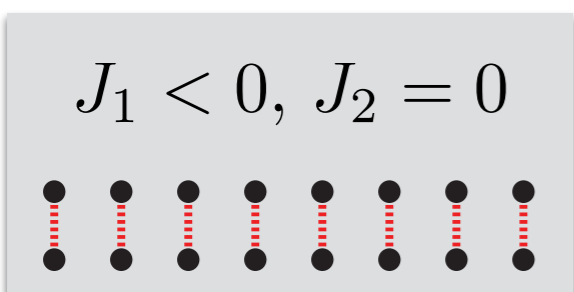
$$Z|A, B\rangle = (-1)^L |A, B\rangle$$

Symmetric MPS



$$\begin{aligned}
 X : \tilde{A}_{\alpha\beta}^{ij} &= \sum_{i',j'} \sigma_{i,i'}^x \sigma_{j,j'}^x A_{\alpha\beta}^{i'j'} = \alpha(x) \left[\hat{\sigma}^x A^{ij} \hat{\sigma}^x \right]_{\alpha\beta} \\
 Z : \tilde{A}_{\alpha\beta}^{ij} &= \sum_{i',j'} \sigma_{i,i'}^z \sigma_{j,j'}^z A_{\alpha\beta}^{i'j'} = \alpha(z) \left[\hat{\sigma}^z A^{ij} \hat{\sigma}^z \right]_{\alpha\beta}
 \end{aligned}$$

$V_A^{-1}(x)$ $V_A(x)$



$$\begin{aligned}
 X : \tilde{B}_{\alpha\beta}^{ij} &= \sum_{i',j'} \sigma_{i,i'}^x \sigma_{j,j'}^x B_{\alpha\beta}^{i'j'} = \alpha(x) \left[1 \cdot B^{ij} \cdot 1 \right]_{\alpha\beta} \\
 Z : \tilde{B}_{\alpha\beta}^{ij} &= \sum_{i',j'} \sigma_{i,i'}^z \sigma_{j,j'}^z B_{\alpha\beta}^{i'j'} = \alpha(z) \left[1 \cdot B^{ij} \cdot 1 \right]_{\alpha\beta}
 \end{aligned}$$

$V_B^{-1}(x)$ $V_B(x)$

Projective Representations

Physical representation of symmetry group:

$$\pi(g_1)\pi(g_2) = \pi(g_1g_2) \quad \text{for } g_1, g_2 \in G$$

Apply $\sum_j [\pi(g)]_{ij} A^j = \gamma(g) V^{-1}(g) \cdot A^i \cdot V(g)$ to show that

$$V(g_1)V(g_2) = \chi(g_1, g_2)V(g_1g_2) \quad \text{for } g_1, g_2 \in G$$

↑
(2-)cocycle (phase)

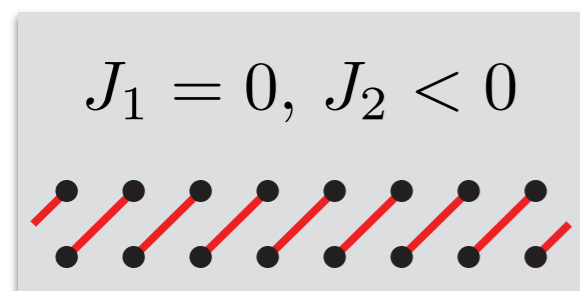
projective
representation

Associativity yields the **cocycle condition**:

$$\chi(g_1, g_2)\chi(g_1g_2, g_3) = \chi(g_2, g_3)\chi(g_1, g_2g_3) \quad \Rightarrow$$

Set of all cocycles:
 $Z^2(G, U(1))$

Projective Representations

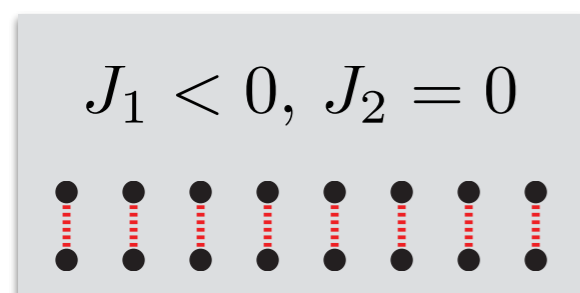


$$\begin{aligned} &\Rightarrow V_A(1) = \mathbb{1} \\ &V_A(x) = \hat{\sigma}^x \\ &V_A(z) = \hat{\sigma}^z \\ &V_A(xz) = \hat{\sigma}^x \hat{\sigma}^z \end{aligned} \quad \Rightarrow$$

$$\begin{aligned} \chi_A(x, z) &= 1 \\ \chi_A(z, x) &= -1 \\ &\vdots \end{aligned}$$

Representations

Cocycles



$$\begin{aligned} &\Rightarrow V_B(1) = 1 \\ &V_B(x) = 1 \\ &V_B(z) = 1 \\ &V_B(xz) = 1 \end{aligned} \quad \Rightarrow$$

$$\chi_B(g_1, g_2) \equiv 1$$

Projective Representations

$$V(z)V(x) = \hat{\sigma}^z \hat{\sigma}^x = -\hat{\sigma}^x \hat{\sigma}^z = -V(xz) = \chi_A(z, x)V(zx)$$

Representations

Cocycles

$$J_1 < 0, J_2 = 0$$


 \Rightarrow

$$V_B(1) = 1$$

$$V_B(x) = 1$$

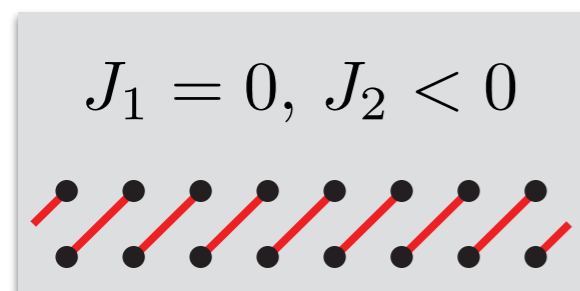
$$V_B(z) = 1$$

$$V_B(xz) = 1$$

 \Rightarrow

$$\chi_B(g_1, g_2) \equiv 1$$

Projective Representations


 \Rightarrow

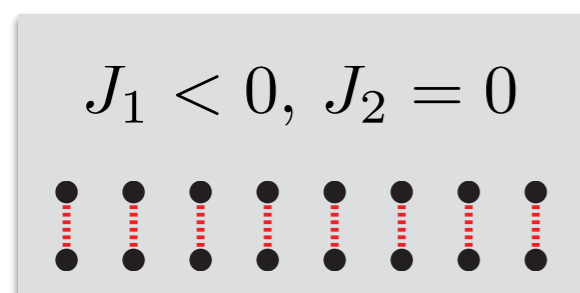
$$\begin{aligned} V_A(1) &= \mathbb{1} \\ V_A(x) &= \hat{\sigma}^x \\ V_A(z) &= \hat{\sigma}^z \\ V_A(xz) &= \hat{\sigma}^x \hat{\sigma}^z \end{aligned}$$

 \Rightarrow

$$\begin{aligned} \chi_A(x, z) &= 1 \\ \chi_A(z, x) &= -1 \\ &\vdots \end{aligned}$$

Representations

Cocycles


 \Rightarrow

$$\begin{aligned} V_B(1) &= 1 \\ V_B(x) &= 1 \\ V_B(z) &= 1 \\ V_B(xz) &= 1 \end{aligned}$$

 \Rightarrow

$$\chi_B(g_1, g_2) \equiv 1$$

Cohomology

There is a „gauge structure“:

$$\begin{aligned} \sum_j [\pi(g)]_{ij} A^j &= \gamma(g) V^{-1}(g) \cdot A^i \cdot V(g) \\ &= \gamma(g) [f(g)V(g)]^{-1} \cdot A^i \cdot [f(g)V(g)] \\ &= \gamma(g) \tilde{V}(g)^{-1} \cdot A^i \cdot \tilde{V}(g) \end{aligned}$$

for arbitrary phases $f(g)$

$$\Rightarrow \tilde{V}(g_1)\tilde{V}(g_2) = \tilde{\chi}(g_1, g_2)\tilde{V}(g_1g_2) \quad (\text{still projective representation})$$

Equivalence relation of cocycles:

$$\tilde{\chi}(g_1, g_2) = \frac{f(g_1)f(g_2)}{f(g_1g_2)} \chi(g_1, g_2) \quad \Leftrightarrow: \quad \tilde{\chi} \sim \chi$$

Cohomology

Set of all cocycles
 $Z^2(G, U(1))$

$$\& \quad \tilde{\chi}(g_1, g_2) = \frac{f(g_1)f(g_2)}{f(g_1g_2)} \chi(g_1, g_2) \quad \Leftrightarrow: \quad \tilde{\chi} \sim \chi$$

\Rightarrow Define the **second cohomology group**

$$H^2(G, U(1)) := Z^2(G, U(1)) / \sim$$

write $[\chi] \in H^2(G, U(1))$

Non-trivial statement:

Two states $|A\rangle$ and $|B\rangle$ belong to the same phase protected by G if and only if $[\chi_A] = [\chi_B] \in H^2(G, U(1))$.

Cohomology

Set of all cocycles
 $Z^2(G, U(1))$

$$\& \quad \tilde{\chi}(g_1, g_2) = \frac{f(g_1)f(g_2)}{f(g_1g_2)} \chi(g_1, g_2) \quad \Leftrightarrow: \quad \tilde{\chi} \sim \chi$$

\Rightarrow Define the **second cohomology group**

$$H^2(G, U(1)) := Z^2(G, U(1)) / \sim$$

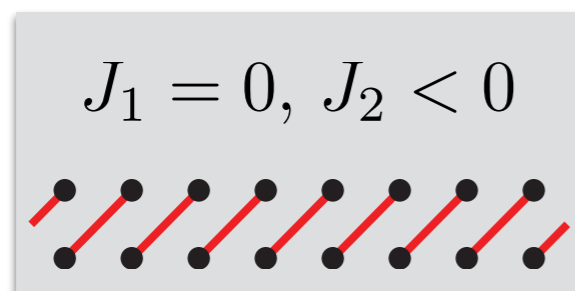
write $[\chi] \in H^2(G, U(1))$

Intuition:

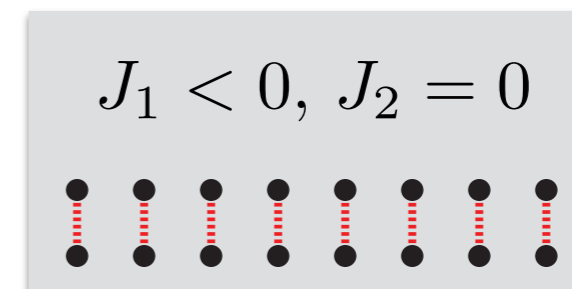
$$\lambda \rightarrow H(\lambda) \rightarrow |A(\lambda)\rangle \rightarrow A^i(\lambda) \rightarrow V_\lambda \rightarrow \chi_\lambda \rightarrow [\chi_\lambda]$$

\Rightarrow Continuous map $\lambda \rightarrow [\chi_\lambda] \Rightarrow$ **Constant** on phases

Cohomology



is in the same phase as
(protected by D_2)



if and only if

$$\chi_A \sim \chi_B \quad \text{or equivalently} \quad [\chi_A] = [\chi_B] \in H^2(D_2, \text{U}(1))$$

Cohomology

Assumption: $\chi_A \sim \chi_B$

$$\Rightarrow \chi_A(g_1, g_2) = \frac{f(g_1)f(g_2)}{f(g_1g_2)} \chi_B(g_1, g_2) = \frac{f(g_1)f(g_2)}{f(g_1g_2)}$$

This leads to a **contradiction**:

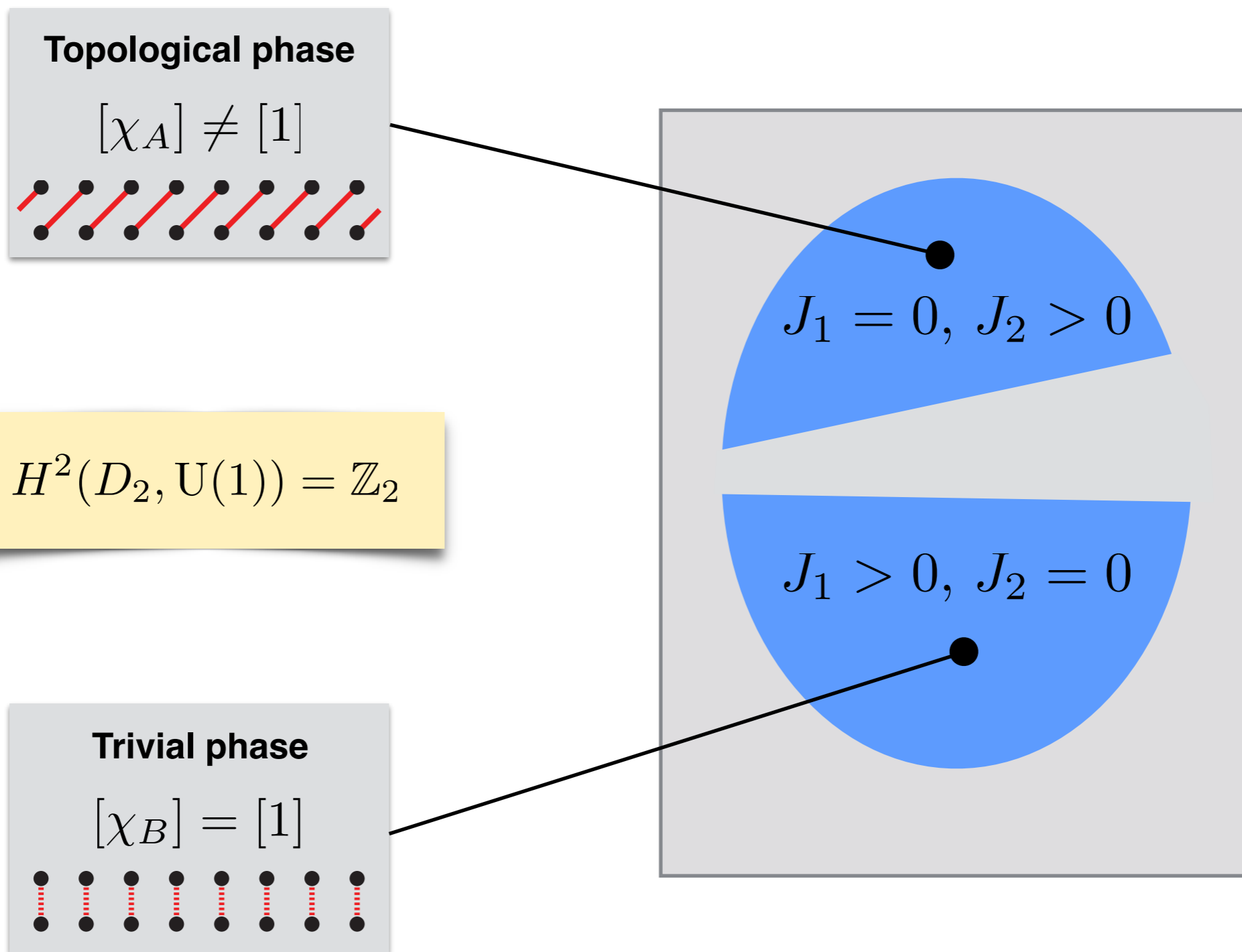
$$+1 = \chi_A(x, z) = \frac{f(x)f(z)}{f(xz)} = \frac{f(z)f(x)}{f(zx)} = \chi_A(z, x) = -1$$

$$\Rightarrow [\chi_A] \neq [\chi_B] = [1]$$

One can show

$$H^2(D_2, U(1)) = \mathbb{Z}_2$$

Cohomology



Edge states

What about chains with **boundaries**?

Non-trivial fact:

One ground state(s), the symmetry acts as $\rho(g) |\Omega\rangle = U_L(g)U_R(g) |\Omega\rangle$

with **projective** boundary operators

\nearrow acts near the **left** \nwarrow **right** edge

$$U_L(g_1)U_L(g_2) = \chi_{\Omega}^*(g_1, g_2) U_L(g_1g_2)$$

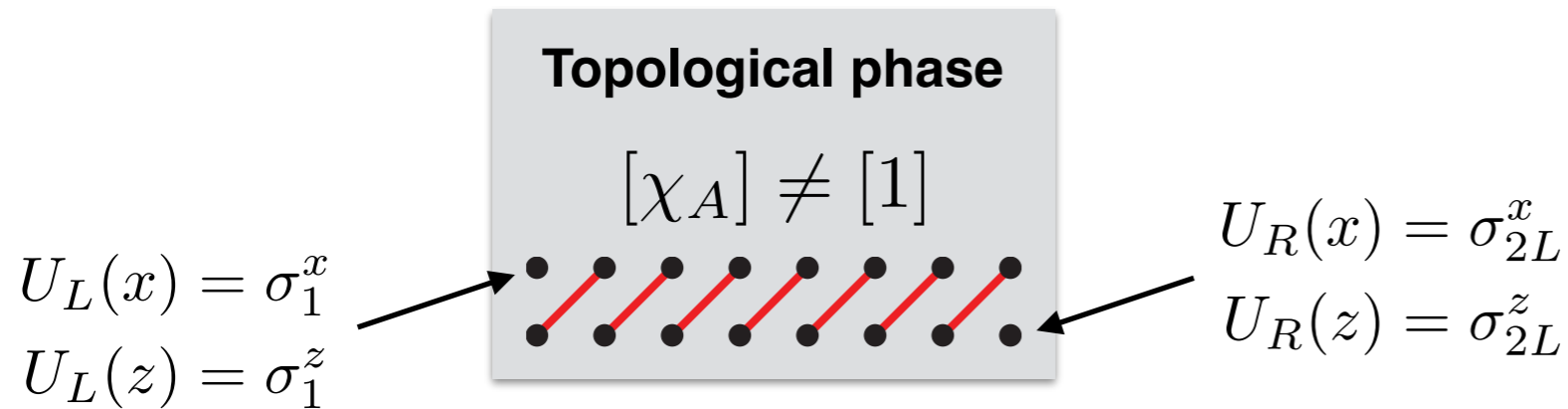
$$U_R(g_1)U_R(g_2) = \chi_{\Omega}(g_1, g_2) U_R(g_1g_2)$$

and $[H, U_L(g)] = 0 = [H, U_R(g)]$ for $L \rightarrow \infty$

(*approximate* symmetries for $L < \infty$)

\Rightarrow Symmetry fractionalization

Edge states

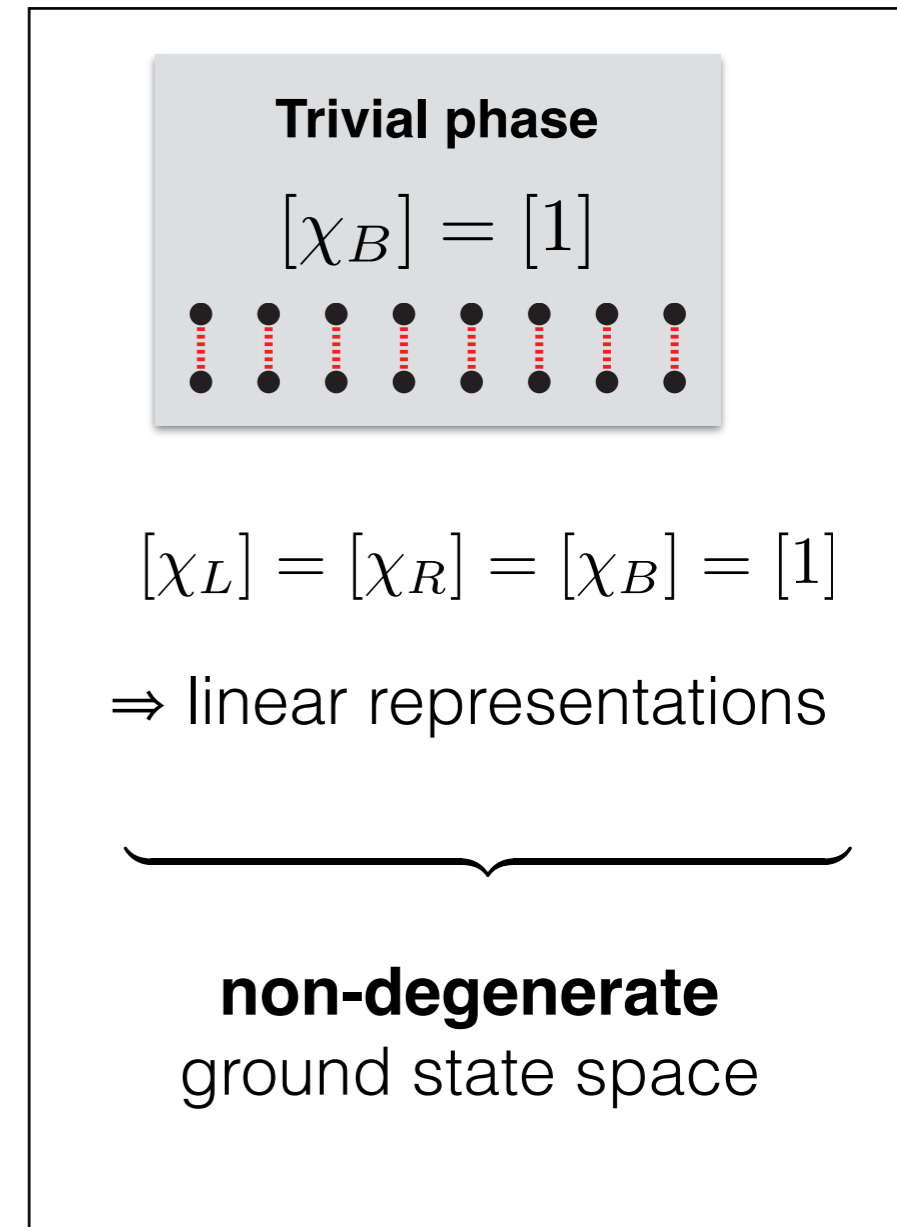


\Rightarrow Non-trivial projective representations of D_2 :

$$[\chi_L] = [\chi_R] = [\chi_A] \neq [1]$$

$$\begin{aligned}
 U_L(x)U_L(z) &= -U_L(z)U_L(x) & & [H, U_L(g)] = 0 \\
 U_R(x)U_R(z) &= -U_R(z)U_R(x) & \& & [H, U_R(g)] = 0
 \end{aligned}$$

4-fold degenerate ground state space



This degeneracy is **robust** against small & symmetric perturbations!

Summary

Theory

- Topological phases
= equivalence classes of gapped Hamiltonians
- No topological phases in one dimension!
- **Symmetry-protected topological (SPT) phases**
= equivalence classes of gapped **symmetric** Hamiltonians
- SPT phases can be labeled/classified by **cohomology classes**
- **Robust** ground state degeneracy for open boundaries
since symmetry acts **projectively** on **boundaries (edge modes)**

Application

The dimerized spin-1/2 chain



⇒ Sebastian Webers talk

is a **SPT phase** with robust **4-fold** ground state degeneracy.